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Integrating out Gluons in Flow Equations

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Abstract

We present an exact nonperturbative flow equation for the average action for quarks which incorporates the effects of gluon fluctuations. With suitable truncations this allows one to compute effective multiquark interactions in dependence on an infrared scale k. Our method amounts to integrating out the gluons with momenta larger than k.

1 Introduction

Effective multiquark interactions have often served as a starting point for QCD-motivated investigations of hadronic physics. A well-known example is the Nambu-Jona-Lasinio model [1] which is based on an effective four-quark interaction. The idea is that these models are valid for quark fluctuations with momenta smaller than a given scale k whereas fluctuations with higher momenta and gluons are thought to be already integrated out. In the more fundamental context of QCD one would like to have some information on the strength of such an interaction in dependence on k. Furthermore, in a realistic theory the effective four-quark interactions will be momentum-dependent. For heavy quarks this momentum dependence translates into the form of the heavy quark potential which is the basis for the non-relativistic quark model for charm and bottom-bound states [2]. Recently a QCD-motivated effective four-quark interaction (at a scale $k \approx 1.5$ GeV) has also been used to derive the emergence of mesonic bound states and chiral symmetry breaking from the solution of nonperturbative flow equations [3]. In order to make direct contact with QCD all these approaches need a method capable of integrating out the gluonic degrees of freedom.

The problem of integrating out completely the gluons seems almost as difficult as the full solution of QCD. Even if an effective infrared cutoff is present in the quark sector the gluons will lead to strong nonlocalities in the quark interactions. Formally, one would have to derive the effective action for gluons and quarks, if needed with an additional infrared cutoff $\sim k$ in the quark sector. From there the gluonic field equations can be derived, giving the (classical) gluon field $A_\mu^{(cl)}$ as a functional of the quark field ψ. Inserting this solution into the effective action then leads to an effective action for the quarks from which the effective multiquark interactions can be extracted. Obviously such a program must become very complicated since due to confinement a purely gluonic description of the gauge degrees of freedom seems not very appropriate in the low momentum range. New non-perturbative effective degrees of freedom are expected to become relevant for very low momenta below the confinement scale Λ.

For many practical purposes, however, the extreme low momentum behaviour of the effective quark interactions is actually not needed. For example, the use of

the effective heavy quark potential for the low lying charmonium and bottonium states essentially needs information about the effective four-quark interaction at momenta $q^2 \gtrsim (300\ \mathrm{MeV})^2$. Since the quark momenta act as an effective infrared cutoff also for the gluon fluctuations[1], we do not expect that confinement effects are dominant in the range of sufficiently large momenta. One may therefore think of using there an approximation with an additional infrared cutoff k both for the gluon and the quark degrees of freedom. This idea is realized by the concept of the average action Γ_k [4] which obtains from integrating out all quantum fluctuations with momenta larger than k. For a given $k > 0$ one does not expect that Γ_k always describes very accurately the n-point functions for momenta smaller than k - in particular the confinement physics in QCD is cut out for large enough k. On the other hand the effective average action can be interpreted as a good approximation to the generating functional for 1PI Green functions with momenta sufficiently larger than k. A possible first approach for a computation of the potential in the range relevant for heavy quarkonia could therefore be attempted by a computation of the quark average action $\Gamma_k[\psi]$ for k somewhat larger than the confinement scale. We will see next how this can be improved by lowering $k \to 0$, provided the momenta in the effective quark n-point functions are sufficiently large.

The dependence of the average action on the scale k is described by an exact non-perturbative flow equation [5]. This functional differential equation needs a truncation in order to become approximately solvable in a non-perturbative context. Typically, one may retain only gluonic two-, three- and four-point functions in the gauge sector. The question arises in what context this type of truncation can lead to meaningful results. First we should point out that such a truncation is not restricted to the range of applicability of perturbation theory. It is general enough to account for a nonperturbative form of the gluon propagator and vertices. Nevertheless, a simple gluonic description is not expected to remain valid once effective non-perturbative degrees of freedom different from the gluons play an important role. (This is partly related to situations where higher n-point functions for the gluons become important.) The range in k where a "gluonic truncation" of the average action can be trusted is limited by this fact. For k larger than the confinement scale the precise description of the confinement mechanism should not be of dominant

[1]We work here in the Euclidean regime where $q^2 > 0$ and quarks are off-shell.

importance. For k as large as a few GeV one expects that a perturbative description of Γ_k becomes valid. It is well conceivable that once k is lowered, there first appears a region where perturbation theory breaks down but a truncation to a few gluon Green functions still remains a valid approximation. Let us now discuss what happens if one tries to follow the flow of the average action beyond this region towards $k \to 0$ within such a "gluonic truncation". One expects that the low momentum behaviour of the n-point functions becomes only poorly described and may even give a qualitatively wrong picture. In contrast, the Green functions with momenta $q^2 \gtrsim (300\ \mathrm{MeV})^2$ should only mildly be affected by the error from the insufficient truncation. Since an infrared cutoff is already present due to the nonvanishing momenta the variation of an additional and smaller infrared cutoff k should not be very relevant anymore. We conclude that a "gluon truncation" can give reliable results even for $k \to 0$ (all fluctuations included), provided the momenta of the Green functions are sufficiently large. In particular, one wonders if a computation of the heavy quark potential $V(R)$ for the range in R relevant for the low-lying bound states may be feasible along these lines.

Let us next turn to the light mesons. The scale for the formation of mesonic bound states was found [3] to be around $k_\varphi \approx 700$ MeV. For $k < k_\varphi$ a mesonic description becomes appropriate [3]. Following the flow equation of an effective quark-meson model [6] for $k < k_\varphi$ one finds that chiral symmetry breaking sets in at $k \approx 450$ MeV. Subsequently the quarks acquire a constituent mass $m_q \approx 350$ MeV from their Yukawa coupling to the chiral symmetry-breaking order parameter. This implies that quarks effectively decouple for $k \lesssim 300$ MeV. Since mesons do not carry colour one may hope that the details of the confinement mechanism are not crucial for an understanding of the meson mass spectrum and decays. This may hold despite the fact that already the formation of light meson bound states is certainly a highly non-perturbative effect. The exchange of many gluons needs to be considered for the description of bound states.

In order to understand this issue more precisely one may imagine for a moment that at some high scale k_i the classical gluon exchange gives rise to an "initial" effective four-quark interaction $\sim \bar{\psi}\gamma^\mu\psi\frac{g^2}{q^2}\bar{\psi}\gamma_\mu\psi$ and the gluons are neglected for $k < k_i$. If one follows the flow equations for the fermionic model for $k < k_i$ the presence of an "initial" four-fermion coupling influences the trajectories and results

for low enough k in a complicated structure of the effective four-quark interaction. In this sense a great part of "multi-gluon exchanges" is already incorporated in the solution of the purely fermionic flow equation. Nonperturbative gluon effects arise even though the gluons do not appear explicitly for $k < k_i$. This is, however, not the whole story: There are also contributions from gluon fluctuations which are not included in the standard flow of an effective fermionic theory. Contributions from those "residual gluon fluctuations" lead to important corrections to the standard fermionic flow equations. They will be computed explicitly in the present paper. Corrections to the flow of the four-quark interaction involve the two-, three- and four-point functions for gluons. In this context the truncation in the gluon sector only enters through the "residual gluon contributions". In an appropriate formulation the "residual gluons" influence only very indirectly the mesons via their effects on the quarks. For $k < 300$ MeV their contributions are suppressed by the quark constituent masses. It is in this type of formulation that the precise description of confinement may not be needed, and even a relatively inaccurate "gluon truncation" may not destroy the reliability of a calculation of meson properties for $k \to 0$.

In order to address the questions raised in this introduction more quantitatively one needs a method which integrates out the gluon fluctuations with momenta $q^2 > k^2$. Formally, it is easy to do so at a given scale k_1. One needs to compute the effective average action for quarks and gluons $\Gamma_{k_1}[\psi, A]$ at this scale. Solving the classical field equation for the gluon field A in dependence on ψ and inserting this classical solution into $\Gamma_{k1}[\psi, A]$ yields exactly an effective average action $\Gamma_{k_1}[\psi]$ involving only the quark fields. The quark-effective action $\Gamma_{k_1}[\psi]$ may then be used as an initial value for solving a pure fermionic flow equation for $k < k_1$. Obviously, the shortcoming of such an approach ist the complete omission of the effects of residual gluon fluctuations with $q^2 < k_1^2$. Choosing a different scale k_2 for eliminating the gluons will lead to a different result. Such a sharp transition between the quark-gluon system and a description involving only quarks necessarily introduces a certain degree of arbritariness.

We propose here a more refined method which changes the classical solution for the gluon field in the course of the evolution towards lower k, thus reflecting the change in the form of $\Gamma_k[\psi, A]$. The result is a smooth procedure for integrating out the gluons, where at every scale k all gluon fluctuations with $q^2 > k^2$ are

included. Nevertheless, we obtain a flow equation for the quark effective average action $\Gamma_k[\psi]$, where gluon fields do not appear explicitly. The inclusion of contributions from residual gluon fluctuations induces correction terms in the flow equation for the quark interactions. In particular, we choose here a formulation where the k-dependent classical solution for A as a functional of ψ includes an effective infrared cutoff $\sim k$. As a consequence, the only nonlocalities in $\Gamma_k[\psi]$ concern length scales shorter than k^{-1}. For the fermionic low momentum modes $\Gamma_k[\psi]$ is an effectively local action. For example, a derivative expansion is meaningful for $q^2 \ll k^2$. The expected nonlocalities arising from the complete elimination of gluon fields (for example a four-quark interaction $\sim \frac{1}{q^2}$ appearing already in the Born approximation) build up only step by step as k is lowered to zero. As a result of this method we will end with an exact nonperturbative flow equation for the scale dependence of $\Gamma_k[\psi]$. Approximations will be needed to solve this equation but they are not limited to perturbative concepts. For example, we may neglect ψ^6 interactions which should be roughly equivalent to neglecting the influence of the baryons on the meson properties. Also the correction terms reflecting the residual gluon fluctuations require limited knowledge about the effective gluon propagator and vertices. It is hoped that rather crude approximations for the gluonic vertices can already lead to satisfactory results.

In sect. 2 we develop our formalism for a simple model of two scalar fields. The flow of the effective average action for one of the scalar fields obtains by integrating out the other scalar field at any scale k. This is generalized to quarks and gluons in sect. 3. In sect. 4 we give a first demonstration how this formalism describes the flow of the two- and four-point function in the effective quark theory. The special case of heavy quarks is addressed in sect. 5. Here we argue that only the evolution of the gauge field propagator is needed in this limit and we compute the corresponding flow equation in one of the appendices. Finally, our conclusions are contained in sect. 6.

2 Reduction of degrees of freedom

In this section we consider for simplicity two types of scalar fields, φ and ψ. We want to develop a formalism how to translate evolution equations for the effective average

action for φ and ψ into corresponding equations involving only ψ. The reader may associate φ with the gluon fields and ψ with the quark fields. Our aim is then the construction of the effective average action for quarks out of the coupled quark-gluon system. This amounts to integrating out the gluonic degrees of freedom represented in the simplified model of this section by φ. We start with the scale-dependent generating functional for the connected Green functions

$$W_k[J,K] = \ln \int D\varphi' D\psi' \exp -\{S[\varphi',\psi'] + \Delta_k^{(\varphi)} S[\varphi'] + \Delta_k^{(\psi)} S[\psi'] - J^\dagger \varphi' - K^\dagger \psi'\} \tag{2.1}$$

Here we denote the degrees of freedom contained in φ' (for example the Fourier modes) by φ'^α and similar for ψ', J and K, with[2]

$$J^\dagger \varphi' = J^*_\alpha \varphi'^\alpha, \quad K^\dagger \psi' = K^*_\beta \psi'^\beta \tag{2.2}$$

We have introduced an infrared cutoff quadratic in the fields

$$\Delta_k^{(\varphi)} S[\varphi'] = \frac{1}{2} \varphi'^\dagger R_k^{(\varphi)} \varphi' \tag{2.3}$$

and similar for ψ. This suppresses the contribution of fluctuations with small momenta $q^2 < k^2$ to the functional integral (2.1). Typically $R_k^{(\varphi)}, R_k^{(\psi)}$ are functions of q^2 with the properties

$$R_k \sim \begin{cases} k^2 & \text{for} \quad q^2 \ll k^2 \\ q^2 \exp(-\frac{q^2}{k^2}) & \text{for} \quad q^2 \gg k^2 \end{cases} \tag{2.4}$$

as, for example,

$$R_k^{(\varphi)} = \frac{Z_k q^2 \exp\left(-\frac{q^2}{k^2}\right)}{1 - \exp\left(-\frac{q^2}{k^2}\right)} \tag{2.5}$$

The effective average action $\Gamma_k[\varphi,\psi]$ is related to the Legendre transform of $W_k[J,K]$

$$\tilde{\Gamma}_k[\varphi,\psi] = -W_k[J,K] + J^\dagger \varphi + K^\dagger \psi \tag{2.6}$$

$$\varphi^\alpha = \frac{\partial W_k}{\partial J^*_\alpha}, \quad \psi^\beta = \frac{\partial W_k}{\partial K^*_\beta} \tag{2.7}$$

by subtracting the infrared cutoff term

$$\Gamma_k[\varphi,\psi] = \tilde{\Gamma}_k[\varphi,\psi] - \Delta_k^{(\varphi)} S[\varphi] - \Delta_k^{(\psi)} S[\psi] \tag{2.8}$$

[2]We use indices α, α' etc. for φ and β, β' etc. for ψ.

For $k \to 0$ the infrared cutoff $\Delta_k S = \Delta_k^{(\varphi)} S + \Delta_k^{(\psi)} S$ vanishes and Γ_0 is the usual generating function for the 1PI Green functions. Using the quadratic form of $\Delta_k S$ it is straightforward to derive an exact non-perturbative evolution equation for the dependence of the effective average action on the scale k ($t = \ln k$) [5]

$$\frac{\partial \Gamma_k}{\partial t} = \frac{1}{2} Tr \left\{ (\tilde{\Gamma}_k^{(2)})^{-1} \frac{\partial R_k}{\partial t} \right\} \tag{2.9}$$

Here $\tilde{\Gamma}_k^{(2)} = \Gamma_k^{(2)} + R_k$ and the inverse propagator $\Gamma_k^{(2)}$ is the second functional derivative of Γ_k with respect to the fields. The matrix $R_k = R_k^{(\varphi)} + R_k^{(\psi)}$ is block diagonal in φ and ψ spaces. The presence of the infrared cutoff R_k in $\tilde{\Gamma}_k^{(2)}$ guarantees infrared finiteness for the momentum integral implied by the trace even in case of massless modes. Ultraviolet finiteness is guaranteed by the exponential decay of $\partial R_k / \partial t$ (2.5). A solution of the flow equation (2.9) interpolates between the classical action for $k \to \infty$ (or k equal to some ultraviolet cutoff Λ) and the effective action for $k \to 0$.

The generating functional for the connected Green functions for ψ obtains from (2.1) for $J = 0$

$$W_k[K] \equiv W_k[J = 0, K] \tag{2.10}$$

Correspondingly, we may introduce an effective action expressed only in terms of ψ

$$\tilde{\Gamma}_k[\psi] = \tilde{\Gamma}_k[\varphi_k[\psi], \psi] \tag{2.11}$$

$$\begin{aligned} \Gamma_k[\psi] &= \tilde{\Gamma}_k[\psi] - \Delta_k^{(\psi)} S[\psi] \\ &= \Gamma_k[\varphi_k[\psi], \psi] + \Delta_k^{(\varphi)} S[\varphi_k[\psi]] \end{aligned} \tag{2.12}$$

by inserting the k-dependent solution of the field equation

$$\frac{\partial \tilde{\Gamma}_k[\varphi, \psi]}{\partial \varphi^\alpha}_{|\varphi_k[\psi]} = 0 \tag{2.13}$$

This defines φ_k as a k-dependent functional of ψ. Since for every scale k the definition (2.6) implies

$$\frac{\partial \tilde{\Gamma}_k[\varphi, \psi]}{\partial \varphi^\alpha} = J_\alpha^*, \quad \frac{\partial \tilde{\Gamma}_k[\varphi, \psi]}{\partial \psi^\beta} = K_\beta^* \tag{2.14}$$

and therefore, for $J = 0$,

$$\frac{d\tilde{\Gamma}_k[\psi]}{d\psi^\beta} = \frac{\partial \tilde{\Gamma}_k[\varphi_k, \psi]}{\partial \psi^\beta} + \frac{\partial \varphi_k^\alpha[\psi]}{\partial \psi^\beta} \frac{\partial \tilde{\Gamma}_k[\varphi, \psi]}{\partial \varphi^\alpha}_{|\varphi = \varphi_k} = K_\beta^* \tag{2.15}$$

it is easy to verify that $\tilde{\Gamma}_k[\psi]$ is the Legendre transform of $W_k[K]$ (2.10). One concludes for $k \to 0$ that $\Gamma_0[\psi]$ is the generating functional for the 1PI Green functions for ψ.

We want to employ the flow equation (2.9) for finding the k-dependence of $\Gamma_k[\psi]$. In addition to the corresponding equation for only one type of fields we have here additional contributions from the k-dependence of $\Delta_k^{(\varphi)} S$ in (2.1). We prefer to keep[3] this additional infrared cutoff for the "gluons" in order to maintain approximate locality of $\Gamma_k[\psi]$ on length scales large compared to k^{-1}. The evolution equation for $\Gamma_k[\psi]$ can now be obtained from a variable transformation which amounts to a shift of φ around $\varphi_k[\psi]$

$$\hat{\varphi}^\alpha = \varphi^\alpha - \varphi_k^\alpha[\psi] \tag{2.16}$$

Using the general formalism of appendix A one finds

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma_k[\psi] &= \frac{1}{2}\left(\Gamma_k^{(2)}[\psi] + R_k^{(\psi)}\right)^{-1\beta}{}_{\beta'}\left\{\left(\frac{\partial R_k^{(\psi)}}{\partial t}\right)^{\beta'}{}_{\beta} + \frac{\partial \varphi^*_{k\alpha'}}{\partial \psi^*_{\beta'}}\left(\frac{\partial R_k^{(\varphi)}}{\partial t}\right)^{\alpha'}{}_{\alpha}\frac{\partial \varphi_k^\alpha}{\partial \psi^\beta}\right\}\\ &+\frac{1}{2}\varphi^*_{k\alpha'}[\psi]\left(\frac{\partial R_k^{(\varphi)}}{\partial t}\right)^{\alpha'}{}_{\alpha}\varphi_k^\alpha[\psi] + \frac{1}{2}\left(\tilde{\Gamma}_k^{(2)}[\varphi=\varphi_k,\psi]\right)^{-1\alpha}{}_{\alpha'}\left(\frac{\partial R_k^{(\varphi)}}{\partial t}\right)^{\alpha'}{}_{\alpha}\end{aligned} \tag{2.17}$$

It is easy to verify that this equation reduces in the limit $R_k^{(\varphi)} = 0$ to the equivalent of eq. (2.9) for fields ψ only. The corrections in the first two terms involve the explicit form of the "classical solution" $\varphi_k[\psi]$. If one is interested in 1PI Green functions for ψ with a given number of external legs one only needs a polynomial expansion of $\varphi_k[\psi]$ up to a given order. For example, the evolution of the term $\sim \psi^4$ in $\Gamma_k[\psi]$ needs the classical solution up to the order ψ^4 if the series $\varphi_k[\psi]$ starts with a term quadratic in ψ. Additional knowledge of the form of $\Gamma_k[\hat{\varphi},\psi]$ beyond its value for $\hat{\varphi} = 0$ is only needed for the last correction term in the form of

$$\left(\hat{\Gamma}_k^{(2)}[0,\psi]\right)^{\alpha'}{}_{\alpha} = \frac{\partial^2 \Gamma_k[\varphi,\psi]}{\partial \hat{\varphi}^*_{\alpha'}\partial\hat{\varphi}^\alpha}{}_{|\hat{\varphi}=0} + \left(R_k^{(\varphi)}\right)^{\alpha'}{}_{\alpha} \tag{2.18}$$

Only the ψ-dependence of the effective $\hat{\varphi}$-propagator plays a role for the study of 1PI functions for ψ.

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[3] A different approach [3] uses separate cutoff scales k and $\tilde{k}$ for the "gluons" φ and "quarks" ψ. Letting $k \to 0$ for fixed $\tilde{k}$ one completely integrates out the gluons. This alternative method is most appropriate for massive fields. Then the mass term acts as an infrared cutoff and prevents the appearance of strong nonlocalities for $q^2 < k^2$

We finally note that the definition of $\Gamma_k[\psi]$ proposed in this paper is not the only possible choice of an effective fermionic action. An alternative formulation is briefly discussed in appendix B.

3 Integrating out the gauge fields

The generalizaton of the discussion of the last section for quarks and gluons follows ref. [7]. All formulae of the last section remain valid for arbitrary bosonic fields if the indices α, β (2.2) include internal indices and Lorentz indices in addition to momentum labels[4]. If ψ is a Grassmann variable as appropriate for fermions the matrix R_k in (2.9) becomes (cf. appendix C)

$$R_k = R_k^{(\varphi)} - R_k^{(\psi)} \tag{3.1}$$

Also ψ^* should be replaced by $\bar{\psi}$ and the index summation over β should involve both ψ and $\bar{\psi}$ separately.[5] For the gauge fields we will choose here a formulation with explicit ghost variables in close analogy, but slightly different from the formulation in ref. [7]. This makes our formulation as close as possible to the language of standard perturbation theory.

We start with the action including a gauge-fixing term in the background gauge and a corresponding action for the anticommuting ghost fields $\xi, \bar{\xi}$

$$\hat{S}[\psi', \xi', a; \bar{A}] = S[\psi', A'] + S_{gf}[a; \bar{A}] + S_{gh}[\xi', a; \bar{A}] \tag{3.2}$$

Here S is a gauge invariant functional of the fermion fields $\psi, \bar{\psi}$ and the gauge field

$$A'_\mu = \bar{A}_\mu + a_\mu. \tag{3.3}$$

The background gauge field $\bar{A}_\mu$ appears in the gauge fixing and ghost terms

$$\begin{aligned} S_{gf} &= \frac{1}{2\alpha} \int d^d x G^*_z G^z && (3.4) \\ G^z &= (D^\mu[\bar{A}])^z{}_y a^y_\mu && (3.5) \\ S_{gh} &= \int d^d x \bar{\xi}'_y (-D^\mu[\bar{A}] D_\mu[\bar{A} + a])^y{}_z \xi'^z && (3.6) \end{aligned}$$

[4] For a complex field φ the sums are also over negative internal indices i with $\varphi^{-i}(q) = \varphi^*_i(q)$.

[5] One may again use negative internal indices for labeling $\bar{\psi}$, i.e. $\psi^{-i}(q) = \bar{\psi}_i(q)$ and use the summations of the preceding section including the negative indices.

Here $D_\mu[\bar{A}]$ is the covariant derivative in the adjoint representation in presence of the background gauge field $\bar{A}$. The generating functional for the connected Green functions is defined as usual

$$W[\eta,\zeta,K;\bar{A}] = \int \mathcal{D}\psi'\mathcal{D}\bar{\psi}'\mathcal{D}\xi'\mathcal{D}\bar{\xi}'\mathcal{D}a \exp -\{\hat{S} - \int d^dx[\bar{\eta}\psi' + \eta\bar{\psi}' + \bar{\zeta}\xi' + \zeta\bar{\xi}' + Ka]\} \tag{3.7}$$

We have introduced here also sources ζ for the ghost fields and note that the source K_z^μ couples to the gauge field fluctuation a_μ^z and therefore transforms homogeneously under gauge transformations as an adjoint tensor. The k-dependent version W_k obtains from W by adding to $\hat{S}$ the infrared cutoff piece

$$\Delta_k S = \Delta_k^{(\psi)} S + \Delta_k^{(A)} S + \Delta_k^{(gh)} S \tag{3.8}$$

Here the fermionic cutoff reads

$$\begin{aligned}\Delta_k^{(\psi)} S &= \left(\frac{1}{2}\right) \bar{\psi}'_{\beta'} (R_k^{(\psi)})^{\beta'}{}_{\beta} \psi'^{\beta} \\ &= \left(\frac{1}{2}\right) \int d^dx \bar{\psi}' Z_{\psi,k}(i\gamma^\mu D_\mu[\bar{A}])\; r_k^{(\psi)}(-D^2[\bar{A}]/k^2)\psi' \end{aligned}\tag{3.9}$$

with D_μ the covariant derivative in the appropriate representation ($D^2 = D_\mu D^\mu$) and $r_k^{(\psi)}$ a dimensionless function. The factor $1/2$ is appropriate for Majorana fermions [8], [3]. For the gauge field cutoff we choose

$$\begin{aligned}\Delta_k^{(A)} S &= \frac{1}{2} a^*_{\alpha'} (R_k)^{\alpha'}{}_{\alpha} a^\alpha \\ &= \frac{1}{2} \int d^dx a_\nu^y \left[\mathcal{D}[\bar{A}]\; r_k^{(A)}\left(\frac{\mathcal{Z}_{A,k}^{-1}\mathcal{D}[\bar{A}]}{k^2}\right)\right]_{\nu z}^{y\mu} a_\mu^z \end{aligned}\tag{3.10}$$

with $\mathcal{D}[\bar{A}]$ an appropriate operator generalizing a covariant Laplacian in the adjoint representation which will be explained below. The matrix $\mathcal{Z}_{A,k}$ accounts for an appropriate wave function renormalization. Finally, we take for the ghosts

$$\begin{aligned}\Delta_k^{(gh)} S &= \bar{\xi}'_{\gamma'} (R_k^{(gh)})^{\gamma'}{}_{\gamma} \xi'^{\gamma} \\ &= \int d^dx \bar{\xi}'_y [Z_{gh,k}\mathcal{D}_s[\bar{A}] r_k^{(gh)}(\mathcal{D}_s[\bar{A}]/k^2)]^y{}_z \xi'^z \end{aligned}\tag{3.11}$$

with $\mathcal{D}_s[\bar{A}] = -D^2[\bar{A}]$ in the adjoint representation. A good choice for the dimensionless function r_k is[6]

$$r_k(y) = \frac{e^{-y}}{1-e^{-y}} \tag{3.12}$$

[6]The function $r_k^{(\psi)}$ may also be chosen differently from (3.12) in order to avoid that R_k diverges for vanishing covariant momenta.

such that

$$\lim_{\mathcal{D}\to 0} R_k = Z_k k^2 \tag{3.13}$$

The k-dependent functions $Z_{\psi,k}, \mathcal{Z}_{A,k}$ and $Z_{gh,k}$ will be adapted to corresponding wave function renormalization constants in the kinetic terms for the fermions, gauge fields and ghosts. In principle, they can depend on the background field $\bar{A}$. The infrared cutoff piece $\Delta_k S$ cuts off all quantum fluctuations with covariant momenta smaller than k in the functional integral defining W_k. For covariant momenta larger than k the infrared cutoff is ineffective and its contribution to the propagator is exponentially suppressed.

Performing a Legendre transform and subtracting the IR cutoff piece again (c.f. (2.6), (2.8)) we arrive at the effective average action $\Gamma_k[\psi, \xi, A, \bar{A}]$, where $A = \bar{A} + \bar{a}$ and $\bar{a}$ is conjugate to K. The dependence of Γ_k on the scale k is described by an exact evolution equation analogous to eq. (2.9), with a negative sign for the contributions $\sim R_k^{(\psi)}$ and $R_k^{(gh)}$. We note that Γ_k only involves terms with an even number of ghost fields due to the symmetry $\bar{\xi}' \to -\bar{\xi}', \xi' \to -\xi'$ of the S_{gh} and $\Delta_k^{(gh)} S$. In consequence, the ghost field equations

$$\frac{\delta \Gamma_k}{\delta \bar{\xi}} = 0, \ \frac{\delta \Gamma_k}{\delta \xi} = 0 \tag{3.14}$$

have always the solution $\bar{\xi} = \xi = 0$. We therefore can extract the propagators and vertices for the physical particles from the effective action for $\bar{\xi} = \xi = 0$:

$$\Gamma_k[\psi, A, \bar{A}] = \Gamma_k[\psi, 0, A, \bar{A}] \tag{3.15}$$

Nevertheless, the evolution equation for $\Gamma_k[\psi, A, \bar{A}]$ obtains a contribution from the variation of the infrared cutoff of the ghost fields as given by

$$\begin{aligned} \frac{\partial}{\partial t}\Gamma_k[\psi, A, \bar{A}] &= \frac{1}{2}\mathrm{Tr}\left\{\left(\frac{\partial}{\partial t}R_k^{(A)}\right)\left(\Gamma_k^{(2)} + R_k\right)^{-1}\right\} \\ &\quad - \left(\frac{1}{2}\right)\mathrm{Tr}\left\{\left(\frac{\partial}{\partial t}R_k^{(\psi)}\right)\left(\Gamma_k^{(2)} + R_k\right)^{-1}\right\} - \varepsilon_k \qquad (3.16) \\ \varepsilon_k &= \mathrm{Tr}\left\{\left(\frac{\partial}{\partial t}R_k^{(gh)}\right)\left(\Gamma_k^{(gh)(2)} + R_k^{(gh)}\right)^{-1}\right\} \qquad (3.17) \end{aligned}$$

Here $\Gamma_k^{(2)} + R_k$ in (3.16) is the matrix of second functional derivatives of $\Gamma_k + \Delta_k^{(\psi)} S + \Delta_k^{(A)} S$ with respect to ψ and A at fixed $\bar{A}$. For this we have exploited that the larger

matrix of second functional derivatives of $(\Gamma_k + \Delta_k S)[\psi, \xi, A, \bar{A}]$ is block diagonal in the (ψ, A) and ξ components for $\xi = 0$. The ghost dependence of $\Gamma_k[\psi, \xi, A, \bar{A}]$ appears in the evolution equation for $\Gamma_k[\psi, A, \bar{A}]$ only through the term ε_k which involves the second functional derivative with respect to the ghost fields $\Gamma_k^{(gh)(2)}$, which is evaluated at $\bar{\xi} = \xi = 0$ and may depend on $\psi, A, \bar{A}$.[7] We will not pay much attention to the detailed form of $\Gamma_k^{(gh)(2)}$ in the present paper and approximate it by its "classical" value (cf. (3.6))

$$\Gamma_k^{(gh)(2)} = -D^\mu[\bar{A}] D_\mu[A]. \tag{3.18}$$

In order to complete the formal setup of our investigation we need to specify the operator $\mathcal{D}$ in eq. (3.10). A good choice is

$$\mathcal{D}[\bar{A}] = \Gamma_k^{(A)(2)}[\bar{A}] \tag{3.19}$$

where $\Gamma_k^{(A)(2)}$ is the second functional derivative of $\Gamma_k[\psi, A, \bar{A}]$ with respect to A for fixed $\bar{A}$ and $\psi = 0$, evaluated at the point $A = \bar{A}$. As in previous formulations [7], the effective average action $\Gamma_k[\psi, A, \bar{A}]$ is gauge invariant with respect to simultaneous gauge transformations of ψ, A and $\bar{A}$.

We can now apply the formalism of the last section in order to "integrate out" the gluon fields A. Derivatives with respect to φ in the preceding section are replaced by derivatives with respect to A at fixed background field $\bar{A}$. In particular, the classical field equation, whose solution is A_k, reads (c.f. (2.13))

$$\frac{\delta \tilde{\Gamma}_k[\psi, A, \bar{A}]}{\delta A_\mu^z(x)}_{|A=A_k} = 0 \tag{3.20}$$

At this point A_k becomes a functional of ψ and $\bar{A}$. Due to the covariance of the field equation (3.20) we may gauge transform[8] any given solution $A_k[\psi, \bar{A}]$ into a corresponding solution $A_k[\psi', \bar{A}'] = A_k'$. In consequence, the effective action

$$\Gamma_k^{eff}[\psi, \bar{A}] = \Gamma_k[\psi, A = A_k[\psi, \bar{A}], \bar{A}] + \Delta_k^{(A)} S[A = A_k[\psi, \bar{A}], \bar{A}] \tag{3.21}$$

[7] In the formulation of ref. [7] the quantity $\Gamma_k^{(gh)(2)}$ is replaced by $D_s[\bar{A}]$. For the computation of the evolution of $\Gamma_k[\psi, A, A]$ performed in ref. [7] this is equivalent to (3.18).

[8] The functional dependence of A_k on ψ and $\bar{A}$ is such that a gauge transformation on ψ and $\bar{A}$ results in a corresponding inhomogeneous gauge transformation of A_k.

is gauge-invariant. For our purposes we want to work with an effective action involving only the quark fields. This requires to fix the background field $\bar{A}$ conveniently. A possible choice is

$$\bar{A} = 0\ ,\ \ A_k[\psi] = A_k[\psi, \bar{A} = 0] \tag{3.22}$$
$$\Gamma_k[\psi] = \Gamma_k^{eff}[\psi, \bar{A} = 0] = \Gamma_k[\psi, A = A_k[\psi], \bar{A} = 0] + \Delta_k^{(A)} S[A = A_k[\psi], \bar{A} = 0].$$

In this version $R_k^{(A)}$ becomes a simple function of momenta. Summarizing our adaptation of eq. (2.17) for quarks and gluons and using the results of appendix C one obtains[9]

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma_k[\psi] \;\; = \;\; & -\left(\frac{1}{2}\right)\left(\Gamma_k^{(2)}[\psi] + R_k^{(\psi)}\right)^{-1\beta}{}_{\beta'}\left\{\left(\frac{\partial R_k^{(\psi)}}{\partial t}\right)^{\beta'}{}_{\beta} - \frac{\partial A^*_{k\alpha'}}{\partial\bar{\psi}_{\beta'}}\left(\frac{\partial R_k^{(A)}}{\partial t}\right)^{\alpha'}{}_{\alpha}\frac{\partial A_k^\alpha}{\partial\psi^\beta}\right\} \\
& +\frac{1}{2}\left(\Gamma_k^{(2)}[\psi, A = A_k[\psi], \bar{A} = 0] + R_k^{(A)}\right)^{-1\alpha}{}_{\alpha'}\left(\frac{\partial R_k^{(A)}}{\partial t}\right)^{\alpha'}{}_{\alpha} \\
& +\frac{1}{2}A^*_{k\alpha'}\left(\frac{\partial R_k^{(A)}}{\partial t}\right)^{\alpha'}{}_{\alpha} A_k^\alpha - \varepsilon_k[\psi, A_k[\psi], \bar{A} = 0]
\end{aligned} \tag{3.23}$$

The factor $(\frac{1}{2})$ in the first term is absent for Dirac spinors. The flow equation (3.23) is the central equation of this paper.

In order to exploit this equation we need $A_k[\psi]$, $\Gamma_k^{(2)}[\psi, A_k, 0]$ and $\varepsilon_k[\psi, A_k, 0]$. Following ref. [7] it is illustrating to divide $\Gamma_k[\psi, A, \bar{A}]$ into a gauge invariant effective action depending only on one gauge field

$$\Gamma_k[\psi, A] = \Gamma_k[\psi, A, \bar{A} = A] \tag{3.24}$$

and a generalized gauge fixing term which vanishes for $\bar{A} = A$

$$\Gamma_k^{gauge}[\psi, A, \bar{A}] = \Gamma_k[\psi, A, \bar{A}] - \Gamma_k[\psi, A]. \tag{3.25}$$

Besides the classical gauge-fixing term and quantum corrections to it the part $\Gamma_k^{\rm gauge}$ contains also the k-dependent counterterms, as for example a k-dependent gluon mass. (In perturbation theory this vanishes for $k \to 0$.) These counterterms are

[9] For general $\bar{A}$ there is also a purely $\bar{A}$ dependent contribution to $\Gamma_k[\psi, A, \bar{A}]$ which arises, for example, from the effective infrared cutoff for the ghost fields [7]. This plays no role for the choice $\bar{A} = 0$.

related to the gauge-invariant kernel $\Gamma_k[\psi, A]$ by generalized Slavnov-Taylor identities [9], [10]. We will not discuss the role of the counterterms very explicitly in the present paper. (It is conceivable that at some later stage it may become advantageous to subtract them for an improved definition of $\Gamma_k[\psi]$.) Using the explicit form of the gauge invariant infrared cutoff [7]

$$\Delta_k^{(A)} S = \frac{1}{2} \int d^d x (A_y^\nu - \bar{A}_y^\nu)(R_k)_{\nu z}^{y\mu}(A_\mu^z - \bar{A}_\mu^z) \tag{3.26}$$

the relevant classical field equation (3.20) for A becomes

$$\frac{\delta \Gamma_k[\psi, A]}{\delta A_\mu^z} + (R_k^{(A)})_{\nu z}^{y\mu}(A_y^\nu - \bar{A}_y^\nu) + \frac{\delta \Gamma_k^{gauge}[\psi, A, \bar{A}]}{\delta A_\mu^z} = 0. \tag{3.27}$$

As a consequence of the gauge invariance of $\Gamma_k[\psi, A]$ the gauge degrees of freedom contained in A only get fixed by the last two terms in eq. (3.27). Similarly, the inverse propagator for the gauge degrees of freedom obtains contributions only from Γ_k^{gauge}

$$\Gamma_k^{(2)}[\psi, A, \bar{A}] = \Gamma_k^{(2)}[\psi, A] + \Gamma_k^{gauge(2)}[\psi, A, \bar{A}] \tag{3.28}$$

since the contribution from $\Gamma_k^{(2)}[\psi, A]$ vanishes for them.

4 Effective quark interactions

The flow equation (3.23) is an exact nonperturbative equation. Its solution for $k \to 0$ contains the full information on all 1PI Green functions for the fields ψ. Its solution, however, will only be approximative and involves a truncation of the general form of $\Gamma_k[\psi, A, \bar{A}]$. For a first illustration of the effective action for quarks $\Gamma_k[\psi]$ we make here the simple ansatz (with ψ representing Dirac spinors)

$$\begin{aligned} \Gamma_k[\psi, A, \bar{A} = 0] = \int d^4 x \Big\{ & i Z_{\psi,k} \bar{\psi} \gamma^\mu \tilde{D}_\mu \psi \\ & + \frac{1}{4} \tilde{Z}_{F,k} F_z^{\mu\nu} F_{\mu\nu}^z + \frac{1}{2} \tilde{Z}_{F,k} (\partial_\mu A^\mu)^2 + \mathcal{L}_k[\psi] \Big\} \end{aligned} \tag{4.1}$$

where $\mathcal{L}_k[\psi]$ is independent of A. Here $F_{\mu\nu}^z$ is the field strength of a nonabelian $SU(N_c)$ gauge theory and quarks are in the fundamental representation, with

$$\tilde{D}_\mu \psi = \partial_\mu \psi - i g_k \tilde{Z}_{F,k}^{1/2} A_\mu^z T_z \psi \tag{4.2}$$

$$F_{\mu\nu}^z = \partial_\mu A_\nu^z - \partial_\nu A_\mu^z + g_k \tilde{Z}_{F,k}^{1/2} f_{wy}{}^z A_\mu^w A_\nu^y \tag{4.3}$$

The k dependence of Γ_k is encoded in the k-dependence of the renormalized non-abelian gauge coupling g_k, the wave function renormalizations $Z_{\psi,k}$, $\tilde{Z}_{F,k}$ and $\mathcal{L}_k[\psi]$. We will discard the k-dependence of the ghost-action and take $\Gamma_{(gh)k} = S_{gh}$ (3.6). With this ansatz an obvious choice for the wave function renormalization in the effective infrared cutoff is $\mathcal{Z}_{A,k} = \tilde{Z}_{F,k}$ and $Z_{gh,k} = 1$. We note that the truncation (4.1) corresponds to a (renormalized) gauge fixing parameter $\alpha_R = 1$. This is convenient for the illustrative purpose of this section since the classical solution is particularly simple. For practical computations we remain more general and suggest the choice $\alpha_R = 0$.

With the truncation (4.1) the field equation (3.10) reads

$$\tilde{Z}_F F^{\mu\nu}_{z\ ;\nu} + g\tilde{Z}_F^{1/2} Z_\psi \bar{\psi}\gamma^\mu T_z \psi - \tilde{Z}_F \partial^2 r_k(-\partial^2) A^\mu_z - \tilde{Z}_F \partial^\mu \partial_\nu A^\nu_z = 0 \tag{4.4}$$

Here r_k is a dimensionless function of $(-\partial^2/k^2)$ reflecting the details of the infrared cutoff ($\mathcal{D} \equiv -\partial^2 \tilde{Z}_F$) and we choose in momentum space (cf. (3.12))

$$r_k = \frac{\exp\left(-\frac{q^2}{k^2}\right)}{1 - \exp\left(-\frac{q^2}{k^2}\right)}. \tag{4.5}$$

Translating to momentum space one obtains (for details see appendix D), with $\tilde{g} = g_k \tilde{Z}_{F,k}^{1/2}$, the field equation

$$\begin{aligned} & -\tilde{g}\frac{Z_\psi}{\tilde{Z}_F}\int \frac{d^4p}{(2\pi)^4}\bar{\psi}(p)\gamma^\mu T_z \psi(p+q) = \qquad (4.6) \\ = \quad & P(q)A^\mu_z(q) + i\tilde{g} f_{zy}{}^{w}\int \frac{d^4p}{(2\pi)^4} A^y_\nu(q-p)\left\{(q^\nu + p^\nu)A^\mu_w(p) - p^\mu A^\nu_w(p)\right\} \\ & + \frac{\tilde{g}^2}{2}(f_{zx}{}^{u} f_{ywu} + f_{zw}{}^{u} f_{yxu})\int \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} A^{y\mu}(q+p-p')A^x_\nu(-p)A^{w\nu}(p') \end{aligned}$$

where

$$P(q) = q^2 + q^2 r_k = \frac{q^2}{1 - \exp(-\frac{q^2}{k^2})}. \tag{4.7}$$

If we are interested in a polynomial expansion of $\Gamma_k[\psi]$ we can solve (4.6) iteratively. In lowest order one finds

$$(A_k^{(0)}(q))^\mu_z = -\tilde{g}\frac{Z_\psi}{\tilde{Z}_F}\int \frac{d^4p}{(2\pi)^4}\bar{\psi}(p)P^{-1}(q)\gamma^\mu T_z \psi(p+q) \tag{4.8}$$

The next term will involve a correction $\sim \psi^4$ and so forth. (The term $\sim \psi^4$ can be found in appendix D (D.15)). Inserting the lowest order classical solution (4.8) into

the action (4.1) and including the infrared cutoff $\Delta_k^{(A)}S$ the effective action for the quarks reads

$$\Gamma_k[\psi] = \int \frac{d^4q}{(2\pi)^4} \left\{ Z_\psi \bar{\psi}(q)\gamma^\mu q_\mu \psi(q) + \tilde{\mathcal{L}}_k[\psi] \right\} + \Gamma_k^{(A)}[\psi] \tag{4.9}$$

with $\tilde{\mathcal{L}}$ the Fourier transform of $\mathcal{L}_k$ and

$$\Gamma_k^{(A)}[\psi] = -\frac{1}{2} Z_{\psi,k}^2 g_k^2 \int \frac{d^4p_1}{(2\pi)^4} \dots \frac{d^4p_4}{(2\pi)^4} (2\pi)^4 \delta(p_1+p_2-p_3-p_4) \frac{1}{P(p_1-p_3)} \mathcal{M} \tag{4.10}$$

$$\mathcal{M}(p_1,p_2,p_3,p_4) = \left\{ \bar{\psi}_a^i(-p_1)\gamma^\mu (T^z)_i{}^j \psi_j^a(-p_3) \right\} \left\{ \bar{\psi}_b^k(p_4)\gamma_\mu (T_z)_k{}^\ell \psi_\ell^b(p_2) \right\} \tag{4.11}$$

The curled brackets indicate contractions over not explicitly written indices (here spinor indices), $i,j,k,\ell = 1...N_c$ are the colour indices and $a,b = 1...N_f$ the flavour indices of the quarks. By an appropriate Fierz transformation and using the identity

$$(T^z)_i{}^j (T_z)_k{}^\ell = \frac{1}{2}\delta_i^\ell \delta_k^j - \frac{1}{2N_c}\delta_i^j \delta_k^\ell \tag{4.12}$$

we can split $\mathcal{M}$ into three terms [3]

$$\begin{aligned}
\mathcal{M} &= \mathcal{M}_\sigma + \mathcal{M}_\rho + \mathcal{M}_p && (4.13)\\
\mathcal{M}_\sigma &= -\frac{1}{2}\left\{ \bar{\psi}_a^i(-p_1)\psi_i^b(p_2) \right\} \left\{ \bar{\psi}_b^j(p_4)\psi_j^a(-p_3) \right\} \\
&\quad +\frac{1}{2}\left\{ \bar{\psi}_a^i(-p_1)\gamma^5\psi_i^b(p_2) \right\} \left\{ \bar{\psi}_b^j(p_4)\gamma^5\psi_j^a(-p_3) \right\} && (4.14)\\
\mathcal{M}_\rho &= \frac{1}{4}\left\{ \bar{\psi}_a^i(-p_1)\gamma_\mu\psi_i^b(p_2 \right\} \left\{ \bar{\psi}_b^j(p_4)\gamma^\mu\psi_j^a(-p_3) \right\} \\
&\quad +\frac{1}{4}\left\{ \bar{\psi}_a^i(-p_1)\gamma_\mu\gamma^5\psi_i^b(p_2) \right\} \left\{ \bar{\psi}_b^j(p_4)\gamma^\mu\gamma^5\psi_j^a(-p_3) \right\} && (4.15)\\
\mathcal{M}_p &= -\frac{1}{2N_c}\left\{ \bar{\psi}_a^i(-p_1)\gamma_\mu\psi_i^a(-p_3) \right\} \left\{ \bar{\psi}_b^j(p_4)\gamma^\mu\psi_j^b(p_2) \right\} && (4.16)
\end{aligned}$$

In terms of the Lorentz invariants

$$\begin{aligned}
s &= (p_1+p_2)^2 = (p_3+p_4)^2 \\
t &= (p_1-p_3)^2 = (p_2-p_4)^2 && (4.17)
\end{aligned}$$

we recognize that the quantum numbers of the fermion bilinears in $\mathcal{M}_\sigma$ correspond to colour singlet, flavour non-singlet scalars in the s-channel and similarly for spin-one mesons for $\mathcal{M}_\rho$. In analogy to ref. [3] we associate these terms with the scalar mesons of the linear σ-model and with the ρ-mesons. The bilinears in the last term

$\mathcal{M}_p$ correspond to a colour and flavour singlet spin-one boson in the t-channel. These are the quantum numbers of the pomeron.

The effective action $\Gamma_{k_0}[\psi]$ (4.9) evaluated at some conveniently chosen scale k_0 constitutes the "initial value" for the subsequent evolution of $\Gamma_k[\psi]$ for $k < k_0$. The evolution of $\Gamma_k[\psi]$ with the scale k is then described by the exact evolution equation (3.23). The latter has to be evaluated with the approximations (4.1). We are interested in the evolution of the two- and four-point functions for the quarks. The respective flow equations for these quantities obtain by taking the second and fourth functional derivative of eq. (3.23) at $\psi = \bar{\psi} = 0$. We label the different contributions on the r.h.s. of eq. (3.6) by

$$\frac{\partial}{\partial t}\Gamma_k[\psi] = -\gamma_\psi + \gamma_{A\psi} + \gamma_A + \gamma_c - \epsilon \tag{4.18}$$

and discuss them separately.

The first term

$$\gamma_\psi = Tr\left\{\left(\Gamma_k^{(2)}[\psi] + R_k^{(\psi)}\right)^{-1}\frac{\partial}{\partial t}R_k^{(\psi)}\right\} \tag{4.19}$$

is the standard contribution of a pure fermionic theory. It is graphically represented by a fermion loop in fig. 1. Including only a four-fermion interaction in $\Gamma_k[\psi]$ the contribution of γ_ψ to the evolution of the two- and four-point functions of the quarks is shown in fig. 2a and fig. 2b, respectively. In appendix E we expand the ψ-dependent propagator $(\Gamma_k^{(2)}[\psi] + R_k^{(\psi)})^{-1}$ in the background quark fields. From there the contributions to the two- and four-point functions are easily computed. In this section we will not elaborate further on this term and rather concentrate on the new "correction terms" which arise from the gluon and ghost fluctuations. More details on γ_ψ can be found in appendix E (cf. (E.14) and (E.17)).

The remaining terms $\gamma_{A\psi}, \gamma_A$ and γ_c involve $R_k^{(A)}$ and reflect the contributions from gluons, whereas ϵ gives the ghost contribution. The term

$$\gamma_A = \frac{1}{2}Tr\left\{\left(\Gamma_k^{(2)} + R_k^{(A)}\right)^{-1}\frac{\partial}{\partial t}R_k^{(A)}\right\} \tag{4.20}$$

involves only a trace over gluonic degrees of freedom and accounts for the contribution of gluon fluctuations around the ψ-dependent classical solution (fig. 3). The contribution of γ_A to the fermionic two- and four-point functions is given by the dependence of $\Gamma_k^{(2)}[\psi, A_k[\psi], 0]^{\alpha}{}_{\alpha'}$ on ψ. Relevant contributions to γ_A therefore arise

from terms in $\Gamma_k[\psi, A, 0]$ which are either quadratic in A and also depend on ψ or are cubic or higher-order in A. In our truncation the first sort of terms is absent and no relevant contribution to γ_A would be present for an abelian gauge theory. For nonabelian gauge theories we get contributions from the three- and four-gluon vertices in $\Gamma_k[\psi, A]$, i.e.

$$\left(\Gamma_k^{(2)}[\psi, A, \bar{A}=0]\right)_{\nu z}^{y\mu} = \tilde{Z}_F \left\{ \left(\tilde{\mathcal{D}}_T[A]\right)_{\nu z}^{y\mu} + \left(\tilde{\mathcal{D}}_L[0]\right)_{\nu z}^{y\mu} - \left(\tilde{\mathcal{D}}_L[A]\right)_{\nu z}^{y\mu} \right\} \delta(x - x') \quad (4.21)$$

with

$$\left(\tilde{\mathcal{D}}_T[A]\right)_{\nu z}^{y\mu} = -(\tilde{D}^2[A])^y{}_z \delta_\nu^\mu + 2i\tilde{g}\,(T_w)^y{}_z\, F^{w\mu}{}_\nu \quad (4.22)$$

$$\left(\tilde{\mathcal{D}}_L[A]\right)_{\nu z}^{y\mu} = -(\tilde{D}_\nu[A])\tilde{D}^\mu[A])^y{}_z \quad (4.23)$$

Here $\tilde{D}_\mu[A]$ represents the covariant derivative in the adjoint representation with gauge coupling $\tilde{g}$ and $F_\nu{}^\mu$ is the nonabelian field strength associated to the gauge field A. We observe that $\Gamma_k[\psi, A, 0]$ is invariant under global gauge transformations of ψ and A. The expression for γ_A does not explicitly depend on ψ in our truncation and $\gamma_A[\psi = 0, A_k]$ or $\Gamma_k[0, A_k, 0]$ cannot contain a term linear in A_k. There is therefore no contribution from γ_A to the flow equation of the fermionic two-point function. An estimate of the contribution to the four-quark interaction from γ_A therefore amounts to a computation of the gluon contribution to the evolution of the term quadratic in A in $\Gamma_k[\psi = 0, A, \bar{A} = 0]$. This is presented in appendix F, and the relevant graphs are indicated in figs. 4a, 4b. Similarly the ghost contribution ϵ (fig. 5) is (with the approximation (3.18)) only a functional of A. The ghost contribution quadratic in A is also evaluated in appendix F, and its contribution $\sim \psi^4$ is depicted in fig. 6.

We have computed $\gamma_A - \epsilon$ in the truncation (4.1). Neglecting a contribution with a different fermionic index structure (the term $\sim \frac{\partial}{\partial t} H_A(q)$ in appendix F) and another contribution $\sim \frac{\partial}{\partial t} \ln \tilde{Z}_F$ from the wave function renormalization $\tilde{Z}_F$ in R_k one obtains (F.56)

$$\begin{aligned}
&\gamma_A - \epsilon = \frac{1}{2} N_c g_k^4 Z_\psi^2 \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_4}{(2\pi)^4} (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) P^{-2}(p_1 - p_3) \\
&\mathcal{G}(p_1 - p_3)\mathcal{M}(p_1, p_2, p_3, p_4) \\
&\mathcal{G}(q) = \frac{k^2}{16\pi^2} \left\{ 4 + 12\frac{k^2}{q^2} - 8\frac{k^4}{(q^2)^2} - \exp\left(-\frac{q^2}{2k^2}\right)\left(9 + 8\frac{k^2}{q^2} - 8\frac{k^4}{(q^2)^2}\right) \right\} \quad (4.24)
\end{aligned}$$

The gluon and ghost fluctuations induce therefore a running of the coefficient function multiplying the term $\sim \mathcal{M}$ in (4.9). More precisely, we may parametrize

$$\Gamma_{k,4}[\psi] = -Z_\psi^2 \int \frac{d^4p_1}{(2\pi)^4}...\frac{d^4p_4}{(2\pi)^4}(2\pi)^4\delta(p_1+p_2-p_3-p_4)$$
$$\{\lambda_m(p_1,p_2,p_3,p_4)\mathcal{M}(p_1,p_2,p_3,p_4) + \quad \text{other index structures}\} \tag{4.25}$$

Then $\gamma_A - \epsilon$ induces for $k < k_o$ a deviation of λ_m from its "classical value"

$$\lambda_m^{(c)}(p_1,p_2,p_3,p_4) = \frac{1}{4}g_k^2\left(P^{-1}(p_1-p_3)+P^{-1}(p_2-p_4)\right) \tag{4.26}$$

as given by the contribution to the flow equation

$$\frac{\partial}{\partial t}\lambda_m[\gamma_A-\epsilon] = -\frac{1}{4}N_c g_k^4\left\{P^{-2}(p_1-p_3)\mathcal{G}(p_1-p_3)+P^{-2}(p_2-p_4)\mathcal{G}(p_2-p_4)\right\} \tag{4.27}$$

Therefore the gluon contribution to $\frac{\partial}{\partial t}\lambda_m$ depends only on t and vanishes for $t \gg k^2$. Only the behaviour of the four-point function for t of the order of k^2 or smaller gets modified. Details of the behaviour of $\mathcal{G}(q)$ for $q^2 \ll k^2$ can be found in appendix F.

We also observe that γ_A and ϵ account for the gluon and ghost contributions to the effective gluon propagator, whereas the contribution from quark loops is implicitly contained in γ_ψ (fig. 2b). We note that the latter is not distinguished any more from any other fermionic contributions, as, for example, from an explicit four-quark interaction in $\mathcal{L}_k[\psi]$. Only in lowest order in standard perturbation theory the contribution from fig. 2b corresponds exactly to the quark contribution to the renormalized gluon propagator.

The contribution γ_c is quadratic in the classical solution $A_k[\psi]$ and gives a contribution to the four-point function

$$\gamma_c = \frac{1}{2}g_k^2 Z_\psi^2 \int \frac{d^4p_1}{(2\pi)^4}...\frac{d^4p_4}{(2\pi)^4}(2\pi)^4\delta(p_1+p_2-p_3-p_4)K(p_1-p_3)\mathcal{M}(p_1,p_2,p_3,p_4) \tag{4.28}$$

with

$$K(q) = q^2P^{-2}(q)\left(\frac{\partial}{\partial t}r_k(q) - \tilde{\eta}_F r_k(q)\right) \tag{4.29}$$

and

$$\tilde{\eta}_F = -\frac{\partial}{\partial t}\ln \tilde{Z}_F \tag{4.30}$$

This again results in a contribution to the running of λ_m

$$\frac{\partial}{\partial t}\lambda_m[\gamma_c] = -\frac{1}{4}g_k^2\left(K(p_1-p_3)+K(p_2-p_4)\right) \tag{4.31}$$

Comparison with eq. (4.26) shows that this contribution accounts exactly for the k-dependence of the infrared cutoff R_k contained in P. Taking only this contribution into acount the solution of the flow equation therefore describes the k-dependence of P^{-1}, whereas $g_{k_0}^2$ is not modified at this level. The effective running of g_k^2 arises through the contributions $-\gamma_\psi + \gamma_A - \epsilon + \gamma_{A\psi}$.

The last contribution $\gamma_{A\psi}$ finally involves the explicit ψ dependence of the classical solution in form of the term $\sim \partial A_k/\partial\psi$. It is graphically represented in fig. 7 and reads

$$\begin{aligned}\gamma_{A\psi} = &-\int \frac{d^4q}{(2\pi)^4}\tilde{Z}_F q^2 \left(\frac{\partial}{\partial t} r_k(q) - \tilde{\eta}_F r_k(q)\right) \\ &\int \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\frac{\delta A^z_{k\mu}(q)}{\delta\psi^a_i(p)}\left(\frac{1}{\Gamma_k^{(2)}[\psi] + R_k^{(\psi)}}\right)^{aj}_{bi}(p,p')\frac{\delta A^\mu_{kz}(-q)}{\delta\bar{\psi}^j_b(p')}\end{aligned} \tag{4.32}$$

In lowest order we insert (4.8) for the classical solution

$$\begin{aligned}\frac{\delta A^z_{k\mu}(q)}{\delta\psi^a_i(p)} &= \tilde{g}\frac{Z_\psi}{\tilde{Z}_F}P^{-1}(q)\bar{\psi}^l_a(p-q)\gamma_\mu(T^z)_l{}^i \\ \frac{\delta A^\mu_{kz}(-q)}{\delta\bar{\psi}^j_b(p')} &= -\tilde{g}\frac{Z_\psi}{\tilde{Z}_F}P^{-1}(q)\gamma^\mu(T_z)_j{}^l\psi^b_l(p'-q)\end{aligned} \tag{4.33}$$

and we expand

$$\Gamma_k^{(2)}[\psi] = \Gamma_k^{(2)}[0] + \Gamma_k^{(4)}[0]\bar{\psi}\psi + ... \tag{4.34}$$

$$\begin{aligned}\left(\Gamma_k^{(2)}[0]\right)^{aj}_{bi} &= Z_\psi \not{q}\,\delta^a_b\delta^j_i(2\pi)^4\delta(q-q') \\ &\quad + (L_k^{(2)}[0])^a_b(q)\delta^j_i(2\pi)^4\delta(q-q')\end{aligned} \tag{4.35}$$

(Here $L_k^{(2)}[0]$ is a possible contribution from $\mathcal{L}_k[\psi]$ in eq. (4.1), as for example a quark mass term.) The precise index structure of $\Gamma_k^{(4)}[0]\bar{\psi}\psi$ can be found in the appendix E (cf. E.4). Expanding $(\Gamma_k^{(2)} + R_k)^{-1}$ and assuming that $L_k^{(2)}[0]$ is flavour-diagonal

$$\left(L_k^{(2)}[0]\right)^a_b(q) = Z_\psi m_a(q)\delta^a_b\gamma^5 \tag{4.36}$$

one obtains the following contribution to the evolution of the two-point function (fig. 7):

$$\begin{aligned}\gamma^{(2)}_{A\psi} = &\frac{N_c^2-1}{N_c}Z_\psi g_k^2 \int\frac{d^4q}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}(q+p)^2\left(\frac{\partial}{\partial t}r_k(q+p) - \tilde{\eta}_F r_k(q+p)\right) \\ &P^{-2}(q+p)\sum_a \bar{\psi}_a(p)\frac{\not{q}(1+r_k^{(\psi)}(q)) - 2m_a(q)\gamma^5}{q^2(1+r_k^{(\psi)}(q))^2 + m_a^2(q)}\psi^a(p)\end{aligned} \tag{4.37}$$

Two similar contributions to the four-point function arise if we include the terms $\sim \bar{\psi}\psi\psi$ in $\partial A_k/\partial\bar{\psi}$ and similar in $\partial A_k/\partial\bar{\psi}$ (fig. 8a). Another contribution to the four-point function arises through the expansion of $\Gamma_k^{(2)}[\psi]$ (4.34) as shown in fig. 8b. We observe that in perturbation theory the graph 8a corresponds to a contribution to the renormalization of the $\bar{\psi}\psi A_k$ vertex (fig. 9a) and similarly the correspondence of the graph 8b is given in fig. 9b.

To summarize, all the contributions to the flow equation (4.18) have a simple well-defined meaning, which can easily be expressed in terms of graphs with rules given in fig. 10. For small gauge coupling there is a one to one correspondence with associated pieces in standard perturbation theory. Our flow equation is, however, not limited to the perturbative regime.

5 Heavy quark approximation

In the limit of infinitely large quark masses our formalism simplifies considerably. If the momenta in the n-point functions remain bounded (and therefore much smaller than the heavy quark mass), we can omit in eq. (3.23) the terms involving the inverse fermion propagator $(\Gamma_k^{(2)}[\psi] + R_k^{(\psi)})^{-1}$. Their contribution is suppressed by inverse powers of the quark masses. In the language of the last section this results to $\gamma_\psi = 0, \gamma_{A\psi} = 0$. The remaining computation amounts to an investigation of the pure gluon theory with static quarks. This is done most easily in the language where the gluon fields are kept explicitly and the relevant effective action is $\Gamma_k[\psi, A, \bar{A} = 0]$. It is instructive, however, to understand at every step the exact equivalence with the effective quark theory developed in the last sections.

If one wants to extract the effective four-quark interaction which encodes the effective heavy quark potential, one needs the k-dependent effective gluon propagator and the effective vertex $\bar{\psi}\psi A$. We first describe here (for arbitrary quark masses) the general framework how an effective four-quark interaction obtains from "gluon exchange" in the formulation where both quark and gluon degrees of freedom are kept explicitly. We then specialize to the heavy quark limit and show the equivalence with the formulation in terms of only quark degrees of freedom. The inverse gluon propagator

$$\left(\Gamma_k^{(2)}[\psi = 0, A = 0, \bar{A} = 0]\right)^{y\mu}_{\nu z}(q) = (G_A(q)\delta^\mu_\nu + H_A(q)q_\nu q^\mu)\delta^y_z \tag{5.1}$$

specifies the term quadratic in A

$$\Gamma_{k,2}^{(A)} = \frac{1}{2}\int \frac{d^4q}{(2\pi)^4} A_y^\nu(-q)(\Gamma_k^{(2)}[\psi=0, A=0, \bar{A}=0])_{\nu z}^{y\mu} A_\mu^z(q) \tag{5.2}$$

whereas the quark-gluon vertex is encoded in

$$\Gamma_k^{(\bar\psi\psi A)} = \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} \bar\psi_a^i(p) G_\psi(p,q)\gamma^\mu (T_z)_i{}^j \psi_j^a(p+q) A_\mu^z(-q) \tag{5.3}$$

Knowledge of G_A, H_A and G_ψ permits to compute the classical solution A_k in order $\bar\psi\psi$

$$(A_k^{(0)}(q))_z^\nu = -S_\mu^\nu(q)\int \frac{d^4p}{(2\pi)^4} \bar\psi_a^i(p) G_\psi(p,q)\gamma^\mu (T_z)_i{}^j \psi_j^a(p+q) \tag{5.4}$$

Here $S = (\Gamma_k^{(2)}[0] + R_k)^{-1}$ is the gluon propagator in presence of the infrared cutoff

$$(R_k^{(A)})_{\nu z}^{y\mu}(q) = (R_k(q)\delta_\nu^\mu + \tilde R_k(q) q_\nu q^\mu)\delta_z^y \tag{5.5}$$

and reads

$$\begin{aligned} S_\mu^\nu(q) = (G_A(q) + R_k(q))^{-1}\Big\{&\delta_\mu^\nu - q^\nu q_\mu (H_A(q) + \tilde R_k(q))\cdot \\ &[G_A(q) + R_k(q) + q^2(H_A(q) + \tilde R_k(q))]^{-1}\Big\} \end{aligned} \tag{5.6}$$

In general, the reduced three-point function $G_\psi(p,q)$ may still involve Dirac matrices (e.g. terms $\sim \gamma^\nu p_\nu$). In the heavy quark limit considered here we can neglect this possibility and treat G_ψ as a scalar function. Inserting the classical solution into (5.2) and (5.3) and accounting for the term $\Delta_k^{(A)} S$ (3.22) yields the effective quark four point function

$$\begin{aligned} \Gamma_{k,4}^{(\psi)} &= -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} S_\mu^\nu(q) G_\psi(p,q) G_\psi(p',-q) \\ &\{\bar\psi_a^i(p)\gamma^\mu (T_z)_i{}^j \psi_j^a(p+q)\}\{\bar\psi_b^k(p')\gamma_\nu (T^z)_k{}^l \psi_l^b(p'-q)\} \\ &= -\frac{1}{2}\int \frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_4}{(2\pi)^4}(2\pi)^4\delta(p_1+p_2-p_3-p_4)\cdot \\ &\{F_1(p_1,p_2,p_3,p_4)\mathcal{M}(p_1,p_2,p_3,p_4) + F_2(p_1,p_2,p_3,p_4)\mathcal{N}(p_1,p_2,p_3,p_4)\} \end{aligned} \tag{5.7}$$

with

$$\begin{aligned} F_1 &= G_\psi(-p_1, p_1-p_3) G_\psi(p_4, p_2-p_4)(G_A(p_1-p_3) + R_k(p_1-p_3))^{-1} \\ F_2 &= G_\psi(-p_1, p_1-p_3) G_\psi(p_4, p_2-p_4)(H_A(p_1-p_3) + \tilde R_k(p_1-p_3)) \\ &\quad (G_A(p_1-p_3) + R_k(p_1-p_3))^{-1}[G_A(p_1-p_3) + R_k(p_1-p_3) \\ &\quad +(p_1-p_3)^2(H_A(p_1-p_3) + \tilde R_k(p_1-p_3))]^{-1} \end{aligned} \tag{5.8}$$

Here $\mathcal{M}$ is given by (4.13) and

$$\begin{aligned} \mathcal{N}(p_1,p_2,p_3,p_4) &= \{\bar\psi_a^i(-p_1)(\not p_1-\not p_3)(T_z)_i{}^j\psi_j^a(-p_3)\} \\ &\quad \{\bar\psi_b^k(p_4)(\not p_2-\not p_4)(T^z)_k{}^l\psi_l^b(p_2)\} \end{aligned} \tag{5.9}$$

We observe that in the heavy quark approximation the coefficients of the quark interactions in the σ,ρ and pomeron channel (4.13) are all given by the same function F_1. For the general gluon propagator discussed in this section there is also an additional four-quark interaction $\sim \mathcal{N}$ which vanishes in the approximation $H_A = \tilde{R}_k = 0$ used in sect. 4. The general quark bilinear is conveniently parametrized by the real functions $Z_\psi(q)$ and $\bar m_a(q)$

$$\Gamma_{k,2}^{(\psi)} = \sum_a \int \frac{d^4q}{(2\pi)^4}\bar\psi_a^i(q)(Z_\psi(q)\gamma^\mu q_\mu + \bar m_a(q)\gamma^5)\psi_i^a(q) \tag{5.10}$$

The k-dependence of the functions G_A, H_A, G_ψ, Z_ψ and $\bar m_a$ relevant for the two- and four-point functions for the quarks can now be studied using the evolution equation (3.16) for $\Gamma_k[\psi, A, \bar A = 0]$.

In the truncation where only the terms (5.2),(5.3) and (5.10) are kept, it is easy to see that the contributons to the k-dependence of G_ψ, Z_ψ and $\bar m_a$ all involve quark propagators. In the heavy quark limit with fixed external momenta they can therefore be neglected and only the k-dependence of $\Gamma_{k,2}^{(A)}$ needs to be considered. For Z_ψ and G_ψ we may take appropriate momentum-independent "short-distance couplings"

$$\begin{aligned} Z_\psi(q) &= 1 \\ G_\psi(p,q) &= \tilde Z_F^{\frac{1}{2}}(m_\psi)g(m_\psi) \end{aligned} \tag{5.11}$$

with renormalized gauge coupling g taken at the scale $k = m_\psi$ and m_ψ the heavy quark mass. We also may identify $k = m_\psi$ with the "ultraviolet cutoff" or the scale where the initial values for the flow equation are specified, i.e.

$$\begin{aligned} \tilde Z_F(m_\psi) &= 1 \\ G_A(q; k = m_\psi) &= q^2 \end{aligned} \tag{5.12}$$

Solving the flow equation for $G_A(q)$ for $k \to 0$ one can compute the heavy quark four-point function for momenta much smaller than the quark mass. For $\alpha_R = 0$

(see appendix F) it is fully determined by

$$\Gamma_{0,4}^{(\psi)} = -\frac{1}{2}g^2(m_\psi)\int\frac{d^4p_1}{(2\pi)^4}...\frac{d^4p_4}{(2\pi)^4}(2\pi)^4\delta(p_1+p_2-p_3-p_4)$$
$$\lim_{k\to 0} G_A^{-1}(p_1-p_3)\left\{\mathcal{M}(p_1,p_2,p_3,p_4)+\frac{1}{(p_1-p_3)^2}\mathcal{N}(p_1,p_2,p_3,p_4)\right\} \quad (5.13)$$

For example, a potential with a Coulomb term and a linear term with string tension λ would correspond in (5.13) to

$$G_A^{-1}(q) = \frac{1}{q^2}+\frac{16\pi\lambda}{g^2(m_\psi)(q^2)^2} \quad (5.14)$$

The flow equation for G_A is computed in appendix F.

Let us next illustrate the equivalence with the language of sects. 3 and 4. Neglecting terms $\sim(\bar\psi\psi)^3$ or higher powers of fermion bilinears, the k-dependence of $\Gamma_k[\psi]$ is easily understood in the heavy quark limit: We insert in

$$\frac{\partial}{\partial t}\Gamma_k[\psi] = \frac{\partial}{\partial t}\Gamma_{k,2}^{(\psi)}[\psi]+\frac{\partial}{\partial t}\Gamma_{k,2}^{(A)}[A]_{|A=A_k}+\frac{\partial}{\partial t}\Gamma_k^{(\bar\psi\psi A)}[\psi,A]_{|A=A_k} \quad (5.15)$$
$$+\frac{\delta\Gamma_{k,2}^{(A)}}{\delta A_\mu^z(q)}\Big|_{A=A_k}\frac{\partial}{\partial t}(A_k^{(0)}(q))_\mu^z+\frac{\delta\Gamma_k^{(\bar\psi\psi A)}}{\delta A_\mu^z(q)}_{|A=A_k}\frac{\partial}{\partial t}(A_k^{(0)}(q))_\mu^z+\frac{\partial}{\partial t}\Delta_k^{(A)}S_{|A=A_k}$$

the identity for the classical solution

$$\frac{\delta\Gamma_k}{\delta A(-q)}_{|A=A_k} = -\frac{\delta\Delta_k^{(A)}S}{\delta A(-q)}_{|A=A_k} = -R_k^{(A)}(q)A_k(q) \quad (5.16)$$

and obtain

$$\frac{\partial}{\partial t}\Gamma_k[\psi] = \frac{\partial}{\partial t}\Gamma_{k,2}^{(\psi)}[\psi]+\frac{\partial}{\partial t}\Gamma_k^{(\bar\psi\psi A)}[\psi,A_k]+\frac{\partial}{\partial t}\Gamma_{k,2}^{(A)}[A_k]+\frac{1}{2}A_k\frac{\partial R_k^{(A)}}{\partial t}A_k \quad (5.17)$$

The first two terms on the r.h.s. of (5.17) do not contribute in the heavy quark limit with fixed external momenta, the third term corresponds exactly to $\gamma_A-\epsilon$ and the last term to γ_c. These remarks generalize, of course, to higher n-point functions for the quarks.

As a side remark it should be noticed that for given vertices and propagators a computation of $\gamma_A-\epsilon+\gamma_c$ as a functional of A_k is independent of the masses of the quarks. It amounts to a computation within the pure gluon theory without any

explicit reference[10] to the fermions. It is therefore valid for arbitrary quark masses. In consequence, the results for $\gamma_A - \epsilon$ reported in appendix F can be used directly for the r.h.s. of the evolution equation (4.18) for arbitrary quark masses. Of course, the expression of A_k as a functional of ψ involves the fermionic part of the average action and therefore depends on the quark gluon couplings. Also, all vertices and propagators on the r.h.s. of the flow equation have to be taken for the full theory with light fermions.

We conclude that the flow equation (3.23) becomes very simple if the quark mass is large compared to all momenta in the Green functions. This static limit is, however, not yet the full answer to physical questions like the decription of heavy quark scattering or the heavy quark potential. For these purposes the momenta $p_1...p_4$ in the heavy quark four-point function have to be continued analytically to Minkowski space and should be taken on-shell, i.e. $p_1^2 = p_2^2 = p_3^2 = p_4^2 = -m_\psi^2$. They can therefore not be considered as small compared to m_ψ. In particular, one needs in (5.8) the three-point functions $G_\psi(p,q)$ for $p^2 = (p+q)^2 = -m_\psi^2$ in addition to $G_A(q)$. The flow of this vertex function can again be computed in the formulation with quarks and gluons. The correspondence in the pure fermionic flow equations is straightforward and results in a flow equation for the four-point function for momenta on mass shell. By Lorentz symmetry $G_\psi(p,q)$ can only depend on the three independent invariants $p^2, (p+q)^2$ and q^2. In consequence, the on-shell vertex is only a function of q^2

$$G_\psi(p,q)_{|p^2=(p+q)^2=-m_\psi^2} \equiv G_\psi(q) \tag{5.18}$$

For on-shell situations we therefore should replace in (5.13) $g^2(m_\psi)G_A^{-1}(q)$ by $G_\psi^2(q)G_A^{-1}(q) = F(q)$ where $F(q) = \lim_{k\to 0} F_k(q) = \lim_{k\to 0} F_1(q)_{|\text{on shell}}$.The relevant flow equation for $F_k(q)$

$$\frac{\partial}{\partial t}F_k^{-1}(q) = \frac{\frac{\partial}{\partial t}G_A(q)}{G_\psi^2(q)} - 2\frac{\frac{\partial}{\partial t}G_\psi(q)G_A(q)}{G_\psi^3(q)} \tag{5.19}$$

involves also the flow equation for $G_\psi(q)$. We also note that for on-shell quarks $(\not{p} + m_\psi\gamma^5)\psi(p) = 0,\ \bar{\psi}(p)(\not{p} + m_\psi\gamma^5) = 0$ and therefore $(\mathcal{N})$ (5.9) vanishes. Heavy quark scattering is entirely described by $F(q)$. The heavy quark potential can be extracted from this function by a three-dimensional Fourier transform.

[10]This can be generalized for a fermion field-dependent gluon propagator if terms $\sim \bar{\psi}\psi A^2$ are included.

6 Conclusions

In this paper we have developed a new formalism of how to integrate out gluons in QCD. The result is an effective action for quarks with an infrared cutoff k - the average action for quark fields $\Gamma_k[\psi]$. Only gluons with momenta $q^2 \gtrsim k^2$ are included in the computation of $\Gamma_k[\psi]$. An exact non-perturbative evolution equation describes the dependence of Γ_k on the infrared cutoff k. Additional gluon fluctuations are included as k is lowered and lead to "residual gluon corrections" to the purely fermionic flow equation. Our treatment permits a smooth transition from a description in terms of quarks and gluons (appropriate for perturbative QCD) to effective models involving only quarks. For many practical purposes one is interested in the limit $k \to 0$ where all quantum fluctuations are (formally) included.

An approximate solution of the flow equations needs truncations of the most general form of Γ_k. This restricts the present range of practical applicability of our formalism. With the present rough truncations in the gluon sector we see two areas where this method may lead to interesting new results: One is colour-neutral strong interaction physcis, in particular the properties of mesons. The other concerns heavy quark physics at intermediate distances or momentum scales, say $q^2 \approx ((0.3 - 1)\ \mathrm{GeV})^2$. On the other hand, present truncations seem inappropriate for a quantitative treatment of confinement. Here the low momentum properties of the "glue" need to be understood and, most probably, additional composite degrees of freedom should be introduced.

This paper is devoted to the development of the new formalism. Sect. 2 is kept rather general and the general field-theoretical setting can be applied to problems of integrating out degrees of freedom in a wide context. In the following sections we focus on QCD. In view of the length of this paper we have not yet addressed here practical applications, which will involve numerical solutions of truncated flow equations. Nevertheless, the necessary analytical computations for this purpose have already largely been carried out and are presented in the appendices: The heavy quark potential involves the flow equation for the gluon propagator which is computed in appendix F. The residual gluon corrections for the flow equations for light quarks need in addition the results of appendix E. With the help of these formulae the way for numerical solutions of the truncated flow equations seems open

and we hope that interesting quantitative results for QCD at intermediate scales and meson physics will emerge.

Appendix A: Field transformations in the flow equation

For a general k-dependent nonlinear variable transformation $\hat{\varphi} = \hat{\varphi}_k[\varphi, \psi]$, $\hat{\psi} = \hat{\psi}_k[\varphi, \psi]$ the flow equation (2.9) transforms into

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma_k[\hat{\varphi}, \hat{\psi}] &= \frac{\partial}{\partial t}\Gamma_{k_{|\varphi,\psi}} - \frac{\partial\Gamma_k[\hat{\varphi}, \hat{\psi}]}{\partial\hat{\varphi}^\alpha}\frac{\partial\hat{\varphi}_k^\alpha}{\partial t}_{|\varphi,\psi} - \frac{\partial\Gamma_k[\hat{\varphi}, \hat{\psi}]}{\partial\hat{\psi}^\beta}\frac{\partial\hat{\psi}_k^\beta}{\partial t}_{|\varphi,\psi} \\
&= \frac{1}{2}Tr\{(\tilde{\Gamma}_k^{(2)})^{-1}\frac{\partial R_k}{\partial t}\} - \frac{\partial\tilde{\Gamma}_k[\hat{\varphi}, \hat{\psi}]}{\partial\hat{\varphi}^\alpha}\frac{\partial\hat{\varphi}_k^\alpha}{\partial t} - \frac{\partial\tilde{\Gamma}_k[\hat{\varphi}, \hat{\psi}]}{\partial\hat{\psi}^\beta}\frac{\partial\hat{\psi}_k^\beta}{\partial t} \\
&+ \frac{\partial\Delta_k^{(\varphi)}S[\hat{\varphi}]}{\partial\hat{\varphi}^\alpha}\frac{\partial\hat{\varphi}_k^\alpha}{\partial t} + \frac{\partial\Delta_k^{(\psi)}S[\hat{\psi}]}{\partial\hat{\psi}^\beta}\frac{\partial\hat{\psi}_k^\beta}{\partial t}
\end{aligned} \tag{A.1}$$

(Here $\varphi = \varphi_k[\hat{\varphi}, \hat{\psi}]$, the partial derivative $\partial\hat{\varphi}_k^\alpha/\partial t$ is taken for fixed φ and ψ and similar for $\partial\hat{\psi}_k^\beta/\partial t$.) The corresponding matrix of second functional derivatives $\tilde{\Gamma}_k^{(2)}$ reads now, with combined fields

$$\sigma^\gamma = (\varphi^\alpha, \psi^\beta), \;\; \hat{\sigma}^\gamma = (\hat{\varphi}^\alpha, \hat{\psi}^\beta), \tag{A.2}$$

$$(\tilde{\Gamma}_k^{(2)})^\gamma{}_\delta = \frac{\partial^2\tilde{\Gamma}_k}{\partial\hat{\sigma}^*_{\gamma'}\partial\hat{\sigma}^{\delta'}}\frac{\partial\hat{\sigma}^*_{\gamma'}}{\partial\sigma^*_\gamma}\frac{\partial\hat{\sigma}^{\delta'}}{\delta\sigma^\delta} + \frac{\partial\tilde{\Gamma}_k}{\partial\hat{\sigma}^{\delta'}}\frac{\partial^2\hat{\sigma}^{\delta'}}{\partial\sigma^*_\gamma\partial\sigma^\delta} \tag{A.3}$$

We can therefore write

$$Tr\{(\tilde{\Gamma}_k^{(2)})^{-1}\frac{\partial R_k}{\partial t}\} = \hat{G}^{\gamma'}{}_{\delta'}\frac{\partial\sigma^*_\delta}{\partial\hat{\sigma}^*_{\delta'}}\left(\frac{\partial R_k}{\partial t}\right)^\delta_\gamma\frac{\partial\sigma^\gamma}{\partial\hat{\sigma}^{\gamma'}} \tag{A.4}$$

with $\hat{G}$ the inverse of the "covariant second functional derivative"

$$\left(\hat{\Gamma}_k^{(2)}\right)^{\gamma'}_{\delta'} = \Gamma_{k;}{}^{\gamma'}_{\delta'} = \frac{\partial^2\tilde{\Gamma}_k[\hat{\sigma}]}{\partial\hat{\sigma}^*_{\gamma'}\partial\hat{\sigma}^{\delta'}} + \omega^{\gamma'}{}_{\delta'}{}^{\eta'}\frac{\partial\tilde{\Gamma}_k[\hat{\sigma}]}{\partial\hat{\sigma}^{\eta'}} \tag{A.5}$$

and "connection"

$$\omega^{\gamma'}{}_{\delta'}{}^{\eta'} = \frac{\partial\sigma^*_\gamma}{\partial\hat{\sigma}^*_{\gamma'}}\frac{\partial^2\hat{\sigma}^{\eta'}}{\partial\sigma^*_\gamma\partial\sigma^\delta}\frac{\partial\sigma^\delta}{\partial\hat{\sigma}^{\delta'}} \tag{A.6}$$

Inserting the special transformation (2.16)

$$\begin{aligned}
\psi &= \psi_k[\hat{\varphi}, \hat{\psi}] = \hat{\psi} \\
\varphi &= \varphi_k[\hat{\varphi}, \hat{\psi}] = \varphi_k[\hat{\psi}] + \hat{\varphi}
\end{aligned} \tag{A.7}$$

one obtains

$$
\begin{aligned}
\frac{\partial}{\partial t}\Gamma_k[\hat\varphi,\psi] =& \frac{1}{2}\left(\hat\Gamma_k^{(2)}\right)^{-1\beta}{}_{\beta'}\left(\frac{\partial R_k^{(\psi)}}{\partial t}\right)^{\beta'}{}_{\beta} + \frac{1}{2}\left(\hat\Gamma_k^{(2)}\right)^{-1\alpha}{}_{\alpha'}\left(\frac{\partial R_k^{(\varphi)}}{\partial t}\right)^{\alpha'}{}_{\alpha} \\
&+\frac{1}{2}\left(\hat\Gamma_k^{(2)}\right)^{-1\alpha}{}_{\beta}\frac{\partial\varphi^*_{k\alpha'}}{\partial\psi^*_\beta}\left(\frac{\partial R_k^{(\varphi)}}{\partial t}\right)^{\alpha'}{}_{\alpha} + \frac{1}{2}\left(\hat\Gamma_k^{(2)}\right)^{-1\beta}{}_{\alpha'}\left(\frac{\partial R_k^{(\varphi)}}{\partial t}\right)^{\alpha'}{}_{\alpha}\frac{\partial\varphi_k^\alpha}{\partial\psi^\beta} \\
&+\frac{1}{2}\left(\hat\Gamma_k^{(2)}\right)^{-1\beta}{}_{\beta'}\frac{\partial\varphi^*_{k\alpha'}}{\partial\psi^*_{\beta'}}\left(\frac{\partial R_k^{(\varphi)}}{\partial t}\right)^{\alpha'}{}_{\alpha}\frac{\partial{\varphi_k}^\alpha}{\partial\psi^\beta} \\
&+\frac{\partial\tilde\Gamma_k[\hat\varphi,\psi]}{\partial\hat\varphi^\alpha}\frac{\partial\varphi_k^\alpha[\psi]}{\partial t} - \frac{\partial\Delta_k^{(\varphi)}S[\hat\varphi,\psi]}{\partial\hat\varphi^\alpha}\frac{\partial\varphi_k^\alpha[\psi]}{\partial t}
\end{aligned}
\tag{A.8}
$$

The elements of the covariant second functional derivative $\hat\Gamma_k^{(2)}$ read

$$
\begin{aligned}
\left(\hat\Gamma_k^{(2)}\right)^{\alpha}{}_{\alpha'} &= \frac{\partial^2\tilde\Gamma_k[\hat\varphi,\psi]}{\partial\hat\varphi^*_\alpha\partial\hat\varphi^{\alpha'}} \\
\left(\hat\Gamma_k^{(2)}\right)^{\alpha}{}_{\beta} &= \frac{\partial^2\tilde\Gamma_k[\hat\varphi,\psi]}{\partial\hat\varphi^*_\alpha\partial\psi^{\beta}}, \quad \left(\hat\Gamma_k^{(2)}\right)^{\beta}{}_{\alpha} = \frac{\partial^2\tilde\Gamma_k[\hat\varphi,\psi]}{\partial\psi^*_\beta\partial\hat\varphi^{\alpha}} \\
\left(\hat\Gamma_k^{(2)}\right)^{\beta}{}_{\beta'} &= \frac{\partial^2\tilde\Gamma_k[\hat\varphi,\psi]}{\partial\psi^*_\beta\partial\psi^{\beta'}} - \frac{\partial^2\varphi_k^\alpha}{\partial\psi^*_\beta\partial\psi^{\beta'}}\frac{\partial\tilde\Gamma_k[\hat\varphi,\psi]}{\partial\hat\varphi^\alpha}
\end{aligned}
\tag{A.9}
$$

We note that eqs. (A.8),(A.9) hold for arbitrary $\varphi_k[\psi]$ in the transformation (A.7). If we use in addition the definition $\varphi_k[\psi]$ as a solution of the field equation (2.13), we have the properties

$$\Gamma_k[\hat\varphi=0,\psi] = \Gamma_k[\psi] - \Delta_k^{(\varphi)}S[\varphi_k[\psi]] \tag{A.10}$$

$$\frac{\partial\tilde\Gamma_k}{\partial\hat\varphi^\alpha}_{|\hat\varphi=0} = 0 \tag{A.11}$$

In the evolution equation for $\frac{\partial}{\partial t}\Gamma_k[\psi] = \frac{\partial}{\partial t}\Gamma_k[0,\psi] + \frac{\partial}{\partial t}\Delta_k^{(\varphi)}S[\varphi_k]$ the covariant second functional derivative $\hat\Gamma_k^{(2)}$ reduces to the simple second functional derivative of $\tilde\Gamma_k$ with respect to $\hat\varphi$ and ψ. Also $\left(\hat\Gamma_k^{(2)}\right)^{\alpha}{}_{\beta}$ and $\left(\hat\Gamma_k^{(2)}\right)^{\beta}{}_{\alpha}$ vanish and $\hat\Gamma_k^{(2)}$ is therefore block-diagonal in the φ and ψ components, implying in turn $\left(\hat\Gamma_k^{(2)}\right)^{-1\alpha}{}_{\beta} = \left(\hat\Gamma_k^{(2)}\right)^{-1\beta}{}_{\alpha} = 0$. The $\psi-\psi$ components of the second functional derivative $\left(\hat\Gamma_k^{(2)}\right)^{\beta'}{}_{\beta}$ are simply related to the second functional derivative of $\Gamma_k[\psi]$

$$\left(\hat\Gamma_k^{(2)}\right)^{\beta'}{}_{\beta} = \left(\hat\Gamma_k^{(2)}[\psi] + R_k^{(\psi)}\right)^{\beta'}{}_{\beta} \tag{A.12}$$

Appendix B: Alternative formulation

Although suggestive, the definition of $\Gamma_k[\psi]$ by eqs. (2.11) and (2.13) is not the only possibility for an effective action for ψ. As an alternative we propose to define $\bar{\varphi}_k[\psi]$ as the solution of the field equation derived from $\Gamma_k[\varphi,\psi]$ instead of $\tilde{\Gamma}_k[\varphi,\psi]$, i.e. we replace (2.12) and (2.13) by

$$\frac{\partial \Gamma_k[\varphi,\psi]}{\partial \varphi^\alpha}_{|\bar{\varphi}_k[\psi]} = 0 \tag{B.1}$$

$$\bar{\Gamma}_k[\psi] = \Gamma_k[\bar{\varphi}_k[\psi],\psi] \tag{B.2}$$

We briefly discuss the modifications of our formalism and compare the properties of $\Gamma_k[\psi]$ and $\bar{\Gamma}_k[\psi]$ at the end of this section. The functional $\bar{\Gamma}_k[\psi]$ is not related to the Legendre transform of $W_k[K]$ (2.10), but rather to $W_k[J,K]$ with J given by $R_k^{(\varphi)}\bar{\varphi}_k[\psi]$. In the limit $k \to 0$ the difference $\Delta_k S$ between $\tilde{\Gamma}_k[\varphi,\psi]$ and $\Gamma_k[\varphi,\psi]$ vanishes. We conclude that $\bar{\Gamma}_0[\psi]$ and $\Gamma_0[\psi]$ coincide

$$\Gamma_0[\psi] = \bar{\Gamma}_0[\psi] \tag{B.3}$$

and both define the generating functional for the 1PI Green functions for ψ. Inserting the classical solution $\bar{\varphi}_k[\psi]$ instead of $\varphi_k[\psi]$ in the evolution equation (A.8), (A.9) leads to three modifications: First, the last two terms in eq. (A.8) are now absent. Second, the derivatives

$$\begin{aligned}
\frac{\partial \tilde{\Gamma}_k}{\partial \hat{\varphi}^*_\alpha}[\hat{\varphi}=0,\psi] &= (R_k^{(\varphi)})^\alpha{}_{\alpha'}\varphi_k^{\alpha'}[\psi] \\
\frac{\partial \tilde{\Gamma}_k}{\partial \hat{\varphi}^\alpha}[\hat{\varphi}=0,\psi] &= \varphi^*_{k\alpha'}[\psi](R_k^{(\varphi)})^{\alpha'}{}_\alpha
\end{aligned} \tag{B.4}$$

induce an additional contribution in $(\hat{\Gamma}_k^{(2)})^\beta{}_{\beta'}$ which now reads

$$\begin{aligned}
(\hat{\Gamma}_k^{(2)})^\beta{}_{\beta'} &= (\bar{\Gamma}_k^{(2)}[\psi] + \check{R}_k[\psi])^\beta{}_{\beta'} \\
(\check{R}_k[\psi])^{\beta'}{}_\beta &= (R_k^{(\psi)})^{\beta'}{}_\beta + \frac{\partial \varphi^*_{k\alpha'}}{\partial \psi^*_{\beta'}}(R_k^{(\varphi)})^{\alpha'}{}_\alpha \frac{\partial \varphi_k^\alpha}{\partial \psi^\beta}
\end{aligned} \tag{B.5}$$

Third, the off-diagonal elements of $\hat{\Gamma}_k^{(2)}$ do not vanish anymore

$$\begin{aligned}
(\hat{\Gamma}_k^{(2)})^\alpha{}_\beta &= (R_k^{(\varphi)})^\alpha{}_{\alpha'}\frac{\partial \varphi_k^{\alpha'}[\psi]}{\partial \psi^\beta} \\
(\hat{\Gamma}_k^{(2)})^\beta{}_\alpha &= \frac{\partial \varphi^*_{k\alpha'}[\psi]}{\partial \psi^*_\beta}(R_k^{(\varphi)})^{\alpha'}{}_\alpha
\end{aligned} \tag{B.6}$$

The role of these modifications can best be understood if we write formally the original evolution equation (2.9) as

$$\frac{\partial}{\partial t}\Gamma_k[\varphi,\psi] = \frac{1}{2}\tilde{\partial}_t \ln\det(\Gamma_k^{(2)} + R_k) \tag{B.7}$$

where $\tilde{\partial}_t$ acts only on k. We can express Γ_k in terms of the new variables $\hat{\sigma}$ (A.2) and write

$$\frac{\partial}{\partial t}\Gamma_k[\hat{\sigma}] = \frac{1}{2}\tilde{\partial}_t \ln\det\left(\frac{\partial\sigma^*_\gamma}{\partial\hat{\sigma}^*_{\gamma'}}\left(\Gamma_k^{(2)} + R_k\right)^\gamma{}_\delta \frac{\partial\sigma^\delta}{\partial\hat{\sigma}^{\delta'}}\right) \tag{B.8}$$

provided the transformation $\hat{\sigma}(\sigma)$ does not explicitly or implicitly involve R_k. This is realized for the definition of $\bar{\varphi}_k$ by (B.1). Inserting $\hat{\varphi} = 0$ and using eq. (B.1), one obtains

$$\frac{\partial}{\partial t}\bar{\Gamma}_k[\psi] = \frac{1}{2}Tr\left\{\frac{\partial\check{R}_k[\psi]}{\partial t}(\check{\Gamma}_k^{(2)}[\psi] + \check{R}_k[\psi])^{-1}\right\} \tag{B.9}$$

in agreement with (A.8), (B.5), (B.6). Here $\check{\Gamma}_k^{(2)}$ and $\check{R}_k$ are matrices with α and β indices, and $\check{\Gamma}_k^{(2)}$ is block diagonal

$$\begin{aligned}
(\check{\Gamma}_k^{(2)})^{\beta'}{}_\beta &= \frac{\partial^2\bar{\Gamma}_k[\psi]}{\partial\psi^*_{\beta'}\partial\psi^\beta} \\
(\check{\Gamma}_k^{(2)})^{\alpha'}{}_\alpha &= (\Gamma_k^{(2)}[\varphi,\psi])^{\alpha'}{}_{\alpha|\varphi=\bar{\varphi}_k[\psi]}
\end{aligned} \tag{B.10}$$

The matrix

$$(\check{R}_k)^{\gamma'}{}_{\delta'} = \frac{\partial\sigma_\gamma}{\partial\hat{\sigma}^*_{\gamma'}}(R_k)^\gamma{}_\delta \frac{\partial\sigma^\delta}{\partial\hat{\sigma}^{\delta'}} \tag{B.11}$$

has diagonal and off-diagonal elements as given by

$$\begin{aligned}
(\check{R}_k)^{\beta'}{}_\beta &= (R_k^{(\psi)})^{\beta'}{}_\beta + \frac{\partial\bar{\varphi}^*_{k\alpha'}}{\partial\psi^*_{\beta'}}(R_k^{(\varphi)})^{\alpha'}{}_\alpha \frac{\partial\bar{\varphi}_k^\alpha}{\partial\psi^\beta} \\
(\check{R}_k)^{\alpha'}{}_\alpha &= (R_k^{(\varphi)})^{\alpha'}{}_\alpha \\
(\check{R}_k)^{\alpha'}{}_\beta &= (R_k^{(\varphi)})^{\alpha'}{}_\alpha \frac{\partial\bar{\varphi}_k^\alpha}{\partial\psi^\beta} \\
(\check{R}_k)^{\beta'}{}_\alpha &= \frac{\partial\bar{\varphi}^*_{k\alpha'}}{\partial\psi^*_{\beta'}}(R_k^{(\varphi)})^{\alpha'}{}_\alpha
\end{aligned} \tag{B.12}$$

Similar as to the evolution equation for $\Gamma_k[\psi]$ (2.17) we need information on the ψ-dependence of $\bar{\varphi}_k$ and on $\tilde{\Gamma}_k^{(2)}[\bar{\varphi}_k,\psi]$ in order to specify the evolution equation (B.9) for $\bar{\Gamma}_k[\psi]$. It is convenient to split $\check{\Gamma}_k^{(2)} + \check{R}_k$ into a block-diagonal and off-diagonal

piece

$$
\begin{aligned}
&\check{\Gamma}_k^{(2)} + \check{R}_k = D_k + \Delta_k \\
&(\Delta_k)^{\alpha'}{}_{\beta} = (R_k^{(\varphi)})^{\alpha'}{}_{\alpha} \frac{\partial \varphi_k^{\alpha}}{\partial \psi_\beta} \\
&(\Delta_k)^{\beta'}{}_{\alpha} = \frac{\partial \varphi^*_{k\alpha'}}{\partial \psi^*_{\beta'}} (R_k^{(\varphi)})^{\alpha'}{}_{\alpha}
\end{aligned}
\tag{B.13}
$$

and to expand in Δ, using iteratively the identity

$$
(D+\Delta)^{-1} = D^{-1} - D^{-1}\Delta(D+\Delta)^{-1} \tag{B.14}
$$

Even powers of Δ contribute only to the block-diagonal piece of $(D+\Delta)^{-1}$, i.e.

$$
\begin{aligned}
H^{-1} &= D^{-1} + D^{-1}\Delta D^{-1}\Delta D^{-1} + D^{-1}\Delta D^{-1}\Delta D^{-1}\Delta D^{-1}\Delta D^{-1} + ... \\
&= (D - \Delta D^{-1}\Delta)^{-1}
\end{aligned}
\tag{B.15}
$$

whereas odd powers of Δ give an off-diagonal contribution.

In summary, we can write the evolution equation (B.9) in the explicit form

$$
\begin{aligned}
&\frac{\partial}{\partial t}\bar{\Gamma}_k[\psi] = \frac{1}{2}\frac{\partial}{\partial t}(R_k^{(\varphi)})^{\alpha}{}_{\alpha'}(H^{-1})^{\alpha'}{}_{\alpha} \\
&+\frac{1}{2}\frac{\partial}{\partial t}(\check{R}_k)^{\beta}{}_{\beta'}(H^{-1})^{\beta'}{}_{\beta} \\
&-\frac{1}{2}\frac{\partial}{\partial t}(\check{R}_k)^{\alpha}{}_{\beta'}(D^{-1})^{\beta'}{}_{\beta}(\check{R}_k)^{\beta}{}_{\alpha'}(H^{-1})^{\alpha'}{}_{\alpha} \\
&-\frac{1}{2}\frac{\partial}{\partial t}(\check{R}_k)^{\beta}{}_{\alpha'}(D^{-1})^{\alpha'}{}_{\alpha}(\check{R}_k)^{\alpha}{}_{\beta'}(H^{-1})^{\beta'}{}_{\beta}
\end{aligned}
\tag{B.16}
$$

with

$$
\begin{aligned}
H^{\alpha}{}_{\alpha'} &= D^{\alpha}{}_{\alpha'} - (\check{R}_k)^{\alpha}{}_{\beta'}(D^{-1})^{\beta'}{}_{\beta}(\check{R}_k)^{\beta}{}_{\alpha'} \\
H^{\beta}{}_{\beta'} &= D^{\beta}{}_{\beta'} - (\check{R}_k)^{\beta}{}_{\alpha'}(D^{-1})^{\alpha'}{}_{\alpha}(\check{R}_k)^{\alpha}{}_{\beta'}
\end{aligned}
\tag{B.17}
$$

and

$$
\begin{aligned}
D^{\alpha}{}_{\alpha'} &= (\Gamma_k^{(2)}[\bar{\varphi}_k[\psi],\psi])^{\alpha}{}_{\alpha'} + (R_k^{(\varphi)})^{\alpha}{}_{\alpha'} \\
D^{\beta}{}_{\beta'} &= (\bar{\Gamma}_k^{(2)}[\psi])^{\beta}{}_{\beta'} + (\check{R}_k)^{\beta}{}_{\beta'}
\end{aligned}
\tag{B.18}
$$

The main difference with the formulation in sect. 2 is the absence of an effective infrared cutoff in the classical solution (B.1) if φ is a massless field. This solution and

therefore the flow equation (B.16) typically involves then nonlocalities of the form $1/q^2$. The effective action $\bar{\Gamma}_k$ (B.2) equals in lowest (classical) approximation the classical approximation to $\bar{\Gamma}_0$ and the scale k appears only in the contributions from quantum fluctuations. In contrast to Γ_k non-local n-point functions are therefore already present for $k > 0$. At first sight this may seem as an advantage of the use of $\bar{\Gamma}_k$ since in the classical approximation the nonlocalities in Γ_k have to build up in the course of the scale evolution instead of being included from the outset. Going beyond the classical approximation there is, however, a considerable price to pay: First, the flow equation (B.16) is more complicated than (2.17). Also at every scale k one has to deal with the nonlocality of the classical solution instead of the essentially local behaviour of Γ_k. For these reasons we will use in the present paper the version (2.17) of the flow equation.

Appendix C: Field transformation for fermionic models

The formalism of Appendix A can easily be generalized for fermions, with a little care for minus signs from the anticommutation of Grassmann variables. The generalization of (A.1) reads for the special case (A.7), (A.10), (A.11)

$$\frac{\partial}{\partial t}\Gamma_k[\psi] = \frac{1}{2} STr\left\{\left(\tilde{\Gamma}_k^{(2)}\right)^{-1}\frac{\partial R_k}{\partial t}\right\} + \frac{\partial}{\partial t}\Delta_k^{(\varphi)} S_{|\varphi_k} \tag{C.1}$$

Here the supertrace STr contains an additional minus sign for the fermionic part of the trace. Using $STr(AB) = STr(BA)$ we can write (with the notations of appendix A)

$$STr\left\{\left(\tilde{\Gamma}_k^{(2)}\right)^{-1}\frac{\partial R_k}{\partial t}\right\} = STr\left\{C^{-1}\frac{\partial \hat{R}_k}{\partial t}\right\} \tag{C.2}$$

with

$$\frac{\partial}{\partial t}\hat{R}^{\alpha}{}_{\beta} = \tilde{\kappa}\frac{\partial \sigma^*_{\alpha'}}{\partial \hat{\sigma}^*_{\alpha}}\left(\frac{\partial R_k}{\partial t}\right)^{\alpha'}{}_{\beta'}\frac{\partial \sigma^{\beta'}}{\partial \hat{\sigma}^{\beta}} \tag{C.3}$$

and

$$C^{\alpha}{}_{\beta} = \tilde{\kappa}\frac{\partial \sigma^*_{\alpha'}}{\partial \hat{\sigma}^*_{\alpha}}\left(\tilde{\Gamma}_k^{(2)}\right)^{\alpha'}{}_{\beta'}\frac{\partial \sigma^{\beta'}}{\partial \hat{\sigma}^{\beta}} \tag{C.4}$$

Here we have introduced for later convenience a factor $\tilde{\kappa} = i^N$ with N the number of mixed bosonic/fermionic derivatives of the type $\partial\hat{\varphi}^*/\partial\bar{\psi}$ or $\partial\varphi/\partial\hat{\psi}$ etc. Employing

(A.11) one has

$$\left(\tilde{\Gamma}_k^{(2)}\right)^{\alpha'}{}_{\beta'} = \kappa \frac{\partial \hat{\sigma}^*_\gamma}{\partial \sigma^*_{\alpha'}} \left(\hat{\Gamma}_k^{(2)}\right)^{\gamma}{}_{\delta} \frac{\partial \hat{\sigma}^\delta}{\partial \sigma^{\beta'}} \tag{C.5}$$

where

$$\left(\hat{\Gamma}_k^{(2)}\right)^{\gamma}{}_{\delta} = \frac{\partial^2 \tilde{\Gamma}_k}{\partial \hat{\sigma}^\delta \partial \hat{\sigma}^*_\gamma}_{|\hat{\varphi}=0} \tag{C.6}$$

is block diagonal in the bosonic and fermionic subspaces and therefore a bosonic quantity. The factor κ arises from the commutation of $(\partial \hat{\sigma}^*_\gamma / \partial \sigma^*_\alpha)$ with $(\partial / \partial \sigma^\beta)$ and obeys

$$\kappa = \begin{cases} -1 & \text{for} \quad \hat{\sigma}^* = \varphi^*, \hat{\sigma} = \varphi, \sigma^* = \bar{\psi}, \sigma = \psi \\ 1 & \text{else} \end{cases} \tag{C.7}$$

where φ is bosonic and ψ fermionic. In consequence one has $\kappa = \tilde{\kappa}$, and we observe $C = \hat{\Gamma}_k^{(2)}$. This yields finally

$$STr \left\{ \left(\tilde{\Gamma}_k^{(2)}\right)^{-1} \frac{\partial R_k}{\partial t} \right\} = STr \left\{ \left(\hat{\Gamma}_k^{(2)}\right)^{-1} \frac{\partial \hat{R}_k}{\partial t} \right\} \tag{C.8}$$

where the bosonic part of $\hat{R}_k$ is simply given by $R_k^{(\varphi)}$ whereas the fermionic part reads

$$\hat{R}_k^{(\psi)} = R_k^{(\psi)} - \frac{\partial \varphi_k^*}{\partial \bar{\psi}} R_k^{(\varphi)} \frac{\partial \varphi_k}{\partial \psi} \tag{C.9}$$

The present formulation was adapted for Majorana spinors where ψ and $\bar{\psi}$ are not independent. For Dirac spinors the factor 1/2 in front of the fermionic trace is absent. This amounts to a multiplication of $R_k^{(\psi)}$ by two. Furthermore, ψ and $\bar{\psi}$ are now independent and (C.9) must be supplemented by a second term where the role of ψ and $\bar{\psi}$ is exchanged. These two identical contributions also multiply effectively the second term in (C.9) by a factor of two. The overall modification for Dirac spinors is a multiplication of $\hat{R}^{(\psi)}$ by a factor two.

Appendix D: Classical field equations and propagators

In this appendix we derive the field equations and the gauge field propagator from the effective average action $\Gamma_k[\psi, A, \bar{A} = 0]$ in the truncation given by (4.1). We

start with the gauge-invariant kinetic term

$$\Gamma^{(F)}_{kin} = \frac{\tilde{Z}_F}{4}\int d^dx F^{\mu\nu}_z F^z_{\mu\nu} \tag{D.1}$$

$$F^z_{\mu\nu} = \partial_\mu A^z_\nu - \partial_\nu A^z_\mu + \tilde{g} f_{wy}{}^z A^w_\mu A^y_\nu \quad , \quad \tilde{g} = g_k \tilde{Z}^{\frac{1}{2}}_{F,k} \tag{D.2}$$

and write it as a sum of terms quadratic, cubic and quartic in A

$$\Gamma^{(F)}_{kin} = \Gamma^{(F)}_{kin,2} + \Gamma^{(F)}_{kin,3} + \Gamma^{(F)}_{kin,4} \tag{D.3}$$

with

$$\begin{aligned}
\Gamma^{(F)}_{kin,2} &= \frac{\tilde{Z}_F}{2}\int d^dx A^\nu_z(\partial_\nu\partial^\mu - \partial^2\delta^\mu_\nu)A^z_\mu && \text{(D.4)}\\
\Gamma^{(F)}_{kin,3} &= \tilde{g}\frac{\tilde{Z}_F}{2} f_{wy}{}^z \int d^dx A^w_\mu A^y_\nu(\partial^\mu A^\nu_z - \partial^\nu A^\mu_z) && \text{(D.5)}\\
\Gamma^{(F)}_{kin,4} &= \tilde{g}^2\frac{\tilde{Z}_F}{4} C^{yz}_{wx}\int d^dx A^w_\mu A^\mu_y A^x_\nu A^\nu_z && \text{(D.6)}
\end{aligned}$$

and

$$C^{yz}_{wx} = f_{wx}{}^u f^{yz}{}_u = -(T^u)_{wx}(T_u)^{yz} \tag{D.7}$$

(For the last equations we use $f_{wx}{}^u = f^u{}_{wx} = i(T^u)_{wx}$). For our purposes it is convenient to work in momentum space where

$$\begin{aligned}
\Gamma^{(F)}_{kin,2} &= \frac{\tilde{Z}_F}{2}\int \frac{d^dq}{(2\pi)^d} A^\nu_z(-q)A^z_\mu(q)(q^2\delta^\nu_\mu - q^\nu q_\mu) && \text{(D.8)}\\
\Gamma^{(F)}_{kin,3} &= i\tilde{g}\frac{\tilde{Z}_F}{2} f_{wy}{}^z \int \frac{d^dp_1}{(2\pi)^d}\frac{d^dp_2}{(2\pi)^d}\frac{d^dp_3}{(2\pi)^d}(2\pi)^d\delta(p_1-p_2-p_3) &&\\
&\quad (p^\mu_1 A^\nu_z(-p_1) - p^\nu_1 A^\mu_z(-p_1))A^w_\mu(p_2)A^y_\nu(p_3) && \text{(D.9)}\\
\Gamma^{(F)}_{kin,4} &= \tilde{g}^2\frac{\tilde{Z}_F}{4} C^{yz}_{wx}\int \frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_4}{(2\pi)^d}(2\pi)^d\delta(p_1+p_2-p_3-p_4) &&\\
&\quad A^w_\mu(-p_1)A^\mu_y(p_3)A^x_\nu(-p_2)A^\nu_z(p_4). && \text{(D.10)}
\end{aligned}$$

Similarly the gauge invariant fermionic kinetic term reads

$$\begin{aligned}
\Gamma^{(\psi)}_{kin} &= Z_\psi\int\frac{d^dq}{(2\pi)^d}\bar{\psi}^i_a(q)\gamma^\mu q_\mu \psi^a_i(q)\\
&\quad +\tilde{g}Z_\psi\int\frac{d^dp}{(2\pi)^d}\frac{d^dp'}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}(2\pi)^d\delta(p-p'+q)\\
&\quad \bar{\psi}^i_a(p)\gamma^\mu(T_z)_i{}^j\psi^a_j(p')A^z_\mu(-q)
\end{aligned} \tag{D.11}$$

The field equation obtains as

$$
\begin{aligned}
&\frac{\delta}{\delta A_\mu^z(-q)}\left(\Gamma_{kin}^{(\psi)}+\Gamma_{kin}^{(F)}+\Gamma_k^{gauge}+\Delta_k S^{(A)}\right)_{|\bar{A}=0}= \\
&\tilde{g}Z_\psi\int\frac{d^dp}{(2\pi)^d}\bar{\psi}_a^i(p)\gamma^\mu(T_z)_i{}^j\psi_j^a(p+q)+\tilde{Z}_F P(q)A_z^\mu(q) \\
&+i\tilde{g}\tilde{Z}_F f_{zy}{}^w\int\frac{d^dp}{(2\pi)^d}A_\nu^y(q-p)\left\{(q^\nu+p^\nu)A_w^\mu(p)-p^\mu A_w^\nu(p)\right\} \\
&+\tilde{g}^2\frac{\tilde{Z}_F}{2}\tilde{C}_{zxyw}\int\frac{d^dp}{(2\pi)^d}\frac{d^dp'}{(2\pi)^d}A^{y\mu}(q+p-p')A_\nu^x(-p)A^{w\nu}(p')=0 \quad \text{(D.12)}
\end{aligned}
$$

where we use the invariant tensor

$$
\tilde{C}_{zxyw}=f_{zx}{}^u f_{ywu}+f_{zw}{}^u f_{yxu} \quad \text{(D.13)}
$$

For an expansion in powers of $\bar{\psi}\psi$ we can solve the field equation (D.12) iteratively

$$
(A_k(q))_z^\mu=(A_k^{(0)}(q))_z^\mu+(A_k^{(1)}(q))_z^\mu+...
$$

with the result

$$
(A_k^{(0)}(q))_z^\mu=-\tilde{g}\frac{Z_\psi}{\tilde{Z}_F}P^{-1}(q)\int\frac{d^dp}{(2\pi)^d}\bar{\psi}_a^i(p)\gamma^\mu(T_z)_i{}^j\psi_j^a(p+q) \quad \text{(D.14)}
$$

$$
\begin{aligned}
&(A_k^{(1)}(q))_z^\mu=-i\tilde{g}P^{-1}(q)f_{zy}{}^w\int\frac{d^dp}{(2\pi)^d}(A_k^{(0)}(q-p))_\nu^y \\
&\left\{(q^\nu+p^\nu)\left(A_k^{(0)}(p)\right)_w^\mu-p^\mu\left(A_k^{(0)}(p)\right)_w^\nu\right\} \\
&=i\tilde{g}^3\frac{Z_\psi^2}{\tilde{Z}_F^2}P^{-1}(q)f_{zy}{}^w\int\frac{d^dp_1}{(2\pi)^d}...\frac{d^dp_4}{(2\pi)^d}(2\pi)^d\delta(q-p_1-p_2+p_3+p_4) \\
&P^{-1}(p_1-p_3)P^{-1}(p_2-p_4)\Big[(p_2-p_4)^\mu\{\bar{\psi}(-p_1)\gamma_\nu T^y\psi(-p_3)\}\{\bar{\psi}(p_4)\gamma^\nu T_w\psi(p_2)\} \\
&+(p_1-p_3-2q)^\nu\{\bar{\psi}(-p_1)\gamma_\nu T^y\psi(-p_3)\}\{\bar{\psi}(p_4)\gamma^\mu T_w\psi(p_2)\}\Big] \quad \text{(D.15)}
\end{aligned}
$$

The effective inverse gauge field propagator $(\Gamma_k^{(2)})_{\nu z}^{y\mu}(q',q)+R_k^{(A)}(q)\delta_z^y\delta_\nu^\mu(2\pi)^d\delta(q-q')$ is related to the second functional derivatives

$$
\begin{aligned}
&\frac{\delta^2}{\delta A_\mu^z(-q)\delta A_y^\nu(q')}\left(\Gamma_{kin,2}^{(F)}+\Gamma_k^{gauge}+\Delta_k S^{(A)}\right)_{|\bar{A}=0} \\
&=\tilde{Z}_F P(q)\delta_z^y\delta_\nu^\mu(2\pi)^d\delta(q-q') \quad \text{(D.16)}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\delta^2\Gamma_{kin,3}^{(F)}}{\delta A_\mu^z(-q)\delta A_y^\nu(q')}=i\tilde{g}\tilde{Z}_F f_z{}^{yw}\left\{(2q-q')_\nu A_w^\mu(q-q')\right. \\
&\left.+(2q'-q)^\mu A_{w\nu}(q-q')-(q+q')_\sigma A_w^\sigma(q-q')\delta_\nu^\mu\right\} \quad \text{(D.17)}
\end{aligned}
$$

$$\frac{\delta^2\Gamma^{(F)}_{kin,4}}{\delta A^z_\mu(-q)\delta A^\nu_y(q')} = \tilde{g}^2\frac{\tilde{Z}_F}{2}\int\frac{d^dp}{(2\pi)^d}\cdot$$
$$\left\{\tilde{C}_{zx}{}^y{}_w\delta^\mu_\nu A^x_\sigma(q-q'+p)A^{w\sigma}(-p)+2\tilde{C}_z{}^y{}_{xw}A^{x\mu}(q-q'+p)A^w_\nu(-p)\right\} \quad \text{(D.18)}$$

It is now straightforward to expand the effective propagator $(\Gamma^{(2)}_k + R^{(A)}_k)^{-1}$ in powers of A:

$$\begin{aligned}
&\left[\left(\Gamma^{(2)}_k + R^{(A)}_k\right)^{-1}\right]^{y\mu}_{\nu z}(q',q) = \tilde{Z}_F^{-1}P(q)^{-1}\delta^y_z\delta^\mu_\nu(2\pi)^d\delta(q-q') \\
&-i\tilde{g}\tilde{Z}_F^{-1}P(q')^{-1}P(q)^{-1}f_z{}^{yw}\cdot \\
&\{(2q-q')_\nu A^\mu_w(q-q')+(2q'-q)^\mu A_{w\nu}(q-q')-(q+q')_\sigma A^\sigma_w(q-q')\delta^\mu_\nu\} \\
&-\frac{1}{2}\tilde{g}^2\tilde{Z}_F^{-1}P(q')^{-1}P(q)^{-1}\int\frac{d^dp}{(2\pi)^d}\cdot \qquad \text{(D.19)}\\
&\left\{\tilde{C}_{zx}{}^y{}_w\delta^\mu_\nu A^x_\sigma(q-q'+p)A^{w\sigma}(-p)+2\tilde{C}_z{}^y{}_{xw}A^{x\mu}(q-q'+p)A^w_\nu(-p)\right\} \\
&-\tilde{g}^2\tilde{Z}_F^{-1}P(q')^{-1}P(q)^{-1}\int\frac{d^dp}{(2\pi)^d}P(p)^{-1}f_{y'}{}^{yw}f_z{}^{y'w'}\cdot \\
&\left\{(2p-q')_\nu A^{\nu'}_w(p-q')+(2q'-p)^{\nu'}A_{w\nu}(p-q')-(p+q')_\sigma A^\sigma_w(p-q')\delta^{\nu'}_\nu\right\}\cdot \\
&\{(2q-p)_{\nu'}A^\mu_{w'}(q-p)+(2p-q)^\mu A_{w'\nu'}(q-p)-(q+p)_\tau A^\tau_{w'}(q-p)\delta^\mu_{\nu'}\}+0(A^3)
\end{aligned}$$

Eqs. (D.14), (D.15), and (D.19) define the quantities needed for γ_A and $\gamma_{A\psi}$ in sect. 4. For easy reference, we also give the vertices

$$\frac{\delta^3\Gamma^{(F)}_{kin,3}}{\delta A^{\mu z}(q_1)\delta A^{\nu y}(q_2)\delta A^{\sigma w}(q_3)} = i\tilde{g}\tilde{Z}_F f_{zyw}(2\pi)^d\delta(q_1+q_2+q_3)$$
$$\{(q_1-q_2)_\sigma\delta_{\mu\nu}+(q_2-q_3)_\mu\delta_{\nu\sigma}+(q_3-q_1)_\nu\delta_{\mu\sigma}\} \quad \text{(D.20)}$$

$$\frac{\delta^3\Gamma^{(F)}_{kin,4}}{\delta A^z_\mu(-q)\delta A^\nu_y(q')\delta A^w_\sigma(-p)} = \tilde{g}^2\tilde{Z}_F\left\{\tilde{C}_{zw}{}^y{}_x\delta^\mu_\nu A^{x\sigma}(p+q-q')\right.$$
$$\left.+\tilde{C}_z{}^y{}_{wx}\delta^{\mu\sigma}A^x_\nu(p+q-q')+\tilde{C}_z{}^y{}_{xw}\delta^\sigma_\nu A^{x\mu}(p+q-q')\right\} \quad \text{(D.21)}$$

$$\frac{\delta^4\Gamma^{(F)}_{kin,4}}{\delta A^z_\mu(-q)\delta A^\nu_y(q')\delta A^w_\sigma(-p)\delta A^\tau_x(p')} = \tilde{g}^2\tilde{Z}_F(2\pi)^d\delta(q+p-q'-p')$$
$$\left\{\tilde{C}_{zw}{}^{yx}\delta^\mu_\nu\delta^\sigma_\tau+\tilde{C}_z{}^y{}_w{}^x\delta^{\mu\sigma}\delta_{\nu\tau}+\tilde{C}_z{}^{yx}{}_w\delta^\sigma_\nu\delta^\mu_\tau\right\} \quad \text{(D.22)}$$

Appendix E: Expansion of the effective action and propagator in fermion bilinears

The effective action for quarks $\Gamma_k[\psi]$ contains an equal number of ψ and $\bar{\psi}$ fields as a consequence of baryon number conservation. It can therefore be expanded in terms of 1PI functions for an even number of fields. Denoting by $a..,i..,\gamma..(\dot{\gamma}..)$ the flavour, colour, and spinor indices of the Dirac spinors $\psi(\bar{\psi})$, one has in a momentum basis

$$\begin{aligned}
\Gamma_k[\psi] &= \Gamma_k^{(0)}[0] + \int \frac{d^dq}{(2\pi)^d}\frac{d^dq'}{(2\pi)^d}\bar{\psi}^{i\dot{\gamma}}_{\ a}(q)\psi^b_{j\gamma}(q')\left(\Gamma_k^{(2)}[0]\right)^{aj\gamma}_{bi\dot{\gamma}}(q,q') \\
&+\frac{1}{4}\int \frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_4}{(2\pi)^d}\bar{\psi}^{i\dot{\gamma}}_a(-p_1)\psi^b_{j\gamma}(p_2)\bar{\psi}^{k\dot{\delta}}_c(p_4)\psi^d_{l\delta}(-p_3) \\
&\left(\Gamma_k^{(4)}[0]\right)^{acjl\gamma\delta}_{bdik\dot{\gamma}\dot{\delta}}(p_1,p_2,p_3,p_4) + 0[(\bar{\psi}\psi)^3]
\end{aligned} \tag{E.1}$$

As a consequence of momentum conservation and global colour symmetry we write

$$\left(\Gamma_k^{(2)}[0]\right)^{aj\gamma}_{bi\dot{\gamma}}(q,q') = \left(G^{(2)}\right)^{a\gamma}_{b\dot{\gamma}}(q)\delta^j_i(2\pi)^d\delta(q-q') \tag{E.2}$$

$$\left(\Gamma_k^{(4)}[0]\right)^{acjl\gamma\delta}_{bdik\dot{\gamma}\dot{\delta}}(p_1,p_2,p_3,p_4) = \left(G^{(4)}\right)^{acjl\gamma\delta}_{bdik\dot{\gamma}\dot{\delta}}(p_1,p_2,p_3,p_4)(2\pi)^d\delta(p_1+p_2-p_3-p_4) \tag{E.3}$$

with

$$\begin{aligned}
(G^{(4)})^{acjl\gamma\delta}_{bdik\dot{\gamma}\dot{\delta}}(p_1,p_2,p_3,p_4) &= -(G^4)^{cajl\gamma\delta}_{bdki\dot{\delta}\dot{\gamma}}(-p_4,p_2,p_3,-p_1) \\
&= -(G^{(4)})^{aclj\delta\gamma}_{dbik\dot{\gamma}\dot{\delta}}(p_1,-p_3,-p_2,p_4) \\
&= (G^{(4)})^{calj\delta\gamma}_{dbki\dot{\delta}\dot{\gamma}}(-p_4,-p_3,-p_2,-p_1)
\end{aligned} \tag{E.4}$$

In the approximation (E.1) the second functional derivatives read

$$\begin{aligned}
\left(\Gamma_k^{(2)}[\psi]\right)^{aj\gamma}_{bi\dot{\gamma}}(q,q') &= \frac{\delta^2\Gamma_k[\psi]}{\delta\psi^b_{j\gamma}(q')\delta\bar{\psi}^{i\dot{\gamma}}_a(q)} = \left(G^{(2)}\right)^{a\gamma}_{b\dot{\gamma}}(q)\delta^j_i(2\pi)^d\delta(q-q') \\
&+\int\frac{d^dp}{(2\pi)^d}\frac{d^dp'}{(2\pi)^d}\bar{\psi}^{k\dot{\delta}}_c(p)\psi^d_{l\delta}(p')\left(G^{(4)}\right)^{acjl\gamma\delta}_{bdik\dot{\gamma}\dot{\delta}}(-q,q',-p',p)(2\pi)^d\delta(q'-q+p'-p)
\end{aligned} \tag{E.5}$$

$$\begin{aligned}
\frac{\delta^2\Gamma_k[\psi]}{\delta\psi^b_{j\gamma}(q)\delta\psi^d_{l\delta}(q')} &= \frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\bar{\psi}^{i\dot{\gamma}}_a(q+q'-p)\bar{\psi}^{k\dot{\delta}}_c(p) \\
&(G^4)^{acjl\gamma\delta}_{bdik\dot{\gamma}\dot{\delta}}(p-q-q',q,-q',p)
\end{aligned} \tag{E.6}$$

$$\frac{\delta^2\Gamma_k[\psi]}{\delta\bar\psi_a^{i\dot\gamma}(q)\delta\bar\psi_c^{k\dot\delta}(q')} = \frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\psi^b_{j\gamma}(p)\psi^d_{l\delta}(-p-q-q') \\ (G^{(4)})^{acjl\gamma\delta}_{bdik\dot\gamma\dot\delta}(-q,p,p+q+q',q') \tag{E.7}$$

The effective propagator (including the infrared cutoff $R_k^{(\psi)}$) can now be expanded correspondingly. We will use here the ansatz[11] for the two-point function

$$(G^{(2)})^a_b(q) = Z_\psi(c_a(q)\not{q} + m_a(q)\bar\gamma)\delta^a_b \tag{E.8}$$

Noting that in the order $(\bar\psi\psi)^2$ one has to take into account also the off-diagonal $(\psi\psi)$ and $(\bar\psi\bar\psi)$ matrix elements in $\Gamma_k^{(2)}$ one finds in this order:

$$\begin{aligned}
&\left[\left(\Gamma_k^{(2)}[\psi]+R_k^{(\psi)}\right)^{-1}\right]^{aj\dot\gamma}_{bi\gamma}(q,q') = Z_\psi^{-1}h^{(a)\ \dot\gamma}_{\ \gamma}(q)\delta^a_b\delta^j_i(2\pi)^d\delta(q-q') \\
&-Z_\psi^{-2}\int\frac{d^dp}{(2\pi)^d}\frac{d^dp'}{(2\pi)^d}h^{(a)\dot\eta}_{\gamma}(q)(G^{(4)})^{acjl\eta\delta}_{bdik\dot\eta\dot\delta}(-q,q',-p',p) \\
&h^{(b)\dot\gamma}_{\eta}(q')\bar\psi_c^{k\dot\delta}(p)\psi^d_{l\delta}(p')(2\pi)^d\delta(q'-q+p'-p) \\
&+Z_\psi^{-3}\int\frac{d^dp_1}{(2\pi)^d}..\frac{d^dp_4}{(2\pi)^d}\sum_e h^{(a)\dot\eta}_{\gamma}(q)\Big\{(G^{(4)})^{acml\eta\delta}_{edik\dot\eta\dot\delta}(-q,q-p_1-p_2,-p_2,-p_1) \\
&h^{(e)\dot\varepsilon}_{\eta}(q-p_1-p_2)(G^{(4)})^{efjq\epsilon\vartheta}_{bgmp\dot\epsilon\dot\vartheta}(-q'+p_3+p_4,q',p_3,p_4) \\
&+\frac{1}{4}\left(G^{(4)}\right)^{aeql\vartheta\delta}_{gdim\dot\eta\dot\varepsilon}(-q,-p_3,-p_2,p_2-p_3-q) \\
&h^{(e)\dot\varepsilon}_{\eta}(p_2-p_3-q)\left(G^{(4)}\right)^{cfjm\varepsilon\eta}_{bekp\dot\delta\dot\vartheta}(p_1,q',q'+p_1-p_4,p_4)\Big\}h^{(b)\dot\gamma}_{\epsilon}(q') \\
&\bar\psi_c^{k\dot\delta}(-p_1)\psi^d_{l\delta}(p_2)\bar\psi_f^{p\dot\vartheta}(p_4)\psi^g_{q\vartheta}(-p_3)(2\pi)^d\delta(q-q'-p_1-p_2+p_3+p_4)
\end{aligned} \tag{E.9}$$

where the cutoff-propagator h reads ($r_k \equiv r_k^{(\psi)}$ in this appendix)

$$h^{(a)\dot\gamma}_{\gamma}(q) = \frac{((c_a(q)+r_k(q))\not{q}+m_a(q)\bar\gamma)_\gamma^{\ \dot\gamma}}{(c_a(q)+r_k(q))^2q^2+m_a^2(q)} \tag{E.10}$$

This yields the following quadratic and quartic pieces in γ_ψ

$$\gamma_\psi^{(2)} = -Z_\psi^{-1}\int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\sum_a(G^{(4)})^{acil\eta\delta}_{adik\dot\eta\dot\delta}(-q,q,-p,p)H^{(a)\dot\eta}_{\eta}(q)\bar\psi_c^{k\dot\delta}(p)\psi^d_{l\delta}(p) \tag{E.11}$$

and

$$\gamma_\psi^{(4)} = Z_\psi^{-2}\int\frac{d^dp_1}{(2\pi)^d}...\frac{d^dp_4}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\sum_{a,e}H^{(a)\dot\eta}_{\epsilon}(q)\Big\{h^{(e)\dot\varepsilon}_{\eta}(q-p_1-p_2)$$

[11]The matrix $\bar\gamma$ generalizes γ^5 to arbitrary d.

$$
\begin{aligned}
&(G^{(4)})^{acml\eta\delta}_{edik\dot{\eta}\dot{\delta}}(-q, q-p_1-p_2, -p_2, -p_1)(G^{(4)})^{efiq\epsilon\vartheta}_{agmp\dot{\epsilon}\dot{\vartheta}}(-q+p_3+p_4, q, p_3, p_4)\\
&+\frac{1}{4}h^{(e)\dot{\varepsilon}}_{\eta}(p_2-p_3-q)\left(G^{(4)}\right)^{aeql\vartheta\delta}_{gdim\dot{\eta}\dot{\varepsilon}}(-q, -p_3, -p_2, p_2-p_3-q)\\
&\left(G^{(4)}\right)^{cfim\varepsilon\eta}_{aekp\dot{\delta}\dot{\vartheta}}(p_1, q, q+p_1-p_4, p_4)\Big\}\\
&\bar{\psi}^{k\dot{\delta}}_{c}(-p_1)\psi^{d}_{l\delta}(p_2)\bar{\psi}^{p\dot{\vartheta}}_{f}(p_4)\psi^{g}_{q\vartheta}(-p_3)(2\pi)^d\delta(p_1+p_2-p_3-p_4)
\end{aligned}
\tag{E.12}
$$

where we use the shorthand ($\eta_\psi = -\frac{\partial}{\partial t}\ln Z_\psi$)

$$
\begin{aligned}
&H^{(a)\dot{\eta}}_{\eta}(q) = h^{(a)\dot{\gamma}}_{\eta}(q)\left(\frac{\partial}{\partial t}r_k(q) - \eta_\psi r_k(q)\right)\not{q}_{\dot{\gamma}}{}^{\gamma}h^{(a)\dot{\eta}}_{\gamma}(q)\\
&= \left(\frac{\partial}{\partial t}r_k(q) - \eta_\psi r_k(q)\right)\left[(c_a(q)+r_k(q))^2q^2 + m_a^2(q)\right]^{-2}.\\
&\left[((c_a(q)+r_k(q))^2q^2 - m_a^2(q))\not{q} + 2m_a(q)(c_a(q)+r_k(q))q^2\bar{\gamma}\right]
\end{aligned}
\tag{E.13}
$$

To be more concrete, we consider the truncation

$$
\begin{aligned}
&\Gamma^{(\psi)}_{k,4} = -Z_\psi^2\int\frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_4}{(2\pi)^d}(2\pi)^d\delta(p_1+p_2-p_3-p_4)\\
&\Big\{\lambda_\sigma(p_1,p_2,p_3,p_4)\mathcal{M}_\sigma + \lambda_\rho(p_1,p_2,p_3,p_4)\mathcal{M}_\rho +\\
&\lambda_p(p_1,p_2,p_3,p_4)\mathcal{M}_p + \lambda_n(p_1,p_2,p_3,p_4)\mathcal{N}\Big\}
\end{aligned}
\tag{E.14}
$$

with $\lambda_\sigma(-p_4,-p_3,-p_2,-p_1) = \lambda_\sigma(p_1,p_2,p_3,p_4)$ and similar for $\lambda_\rho, \lambda_p, \lambda_n$. This yields

$$
\begin{aligned}
&Z_\psi^{-2}(G^{(4)})^{acjl\gamma\delta}_{bdik\dot{\gamma}\dot{\delta}}(p_1,p_2,p_3,p_4) =\\
&\lambda_\sigma(p_1,p_2,p_3,p_4)\delta^a_d\delta^c_b\delta^j_i\delta^l_k\left(\delta^\gamma_{\dot{\gamma}}\delta^\delta_{\dot{\delta}} - (\bar{\gamma})_{\dot{\gamma}}{}^{\gamma}(\bar{\gamma})_{\dot{\delta}}{}^{\delta}\right)\\
&-\lambda_\sigma(p_1,-p_3,-p_2,p_4)\delta^a_b\delta^c_d\delta^l_i\delta^j_k\left(\delta^\delta_{\dot{\gamma}}\delta^\gamma_{\dot{\delta}} - (\bar{\gamma})_{\dot{\gamma}}{}^{\delta}(\bar{\gamma})_{\dot{\delta}}{}^{\gamma}\right)\\
&-\frac{1}{2}\lambda_\rho(p_1,p_2,p_3,p_4)\delta^a_d\delta^c_b\delta^j_i\delta^l_k\left((\gamma_\mu)_{\dot{\gamma}}{}^{\gamma}(\gamma^\mu)_{\dot{\delta}}{}^{\delta} + (\gamma_\mu\bar{\gamma})_{\dot{\gamma}}{}^{\gamma}(\gamma^\mu\bar{\gamma})_{\dot{\delta}}{}^{\delta}\right)\\
&+\frac{1}{2}\lambda_\rho(p_1,-p_3,-p_2,p_4)\delta^a_b\delta^c_d\delta^l_i\delta^j_k\left((\gamma_\mu)_{\dot{\gamma}}{}^{\delta}(\gamma^\mu)_{\dot{\delta}}{}^{\gamma} + (\gamma_\mu\bar{\gamma})_{\dot{\gamma}}{}^{\delta}(\gamma^\mu\bar{\gamma})_{\dot{\delta}}{}^{\gamma}\right)\\
&-\frac{1}{N_c}\lambda_p(p_1,p_2,p_3,p_4)\delta^a_d\delta^c_b\delta^l_i\delta^j_k(\gamma_\mu)_{\dot{\gamma}}{}^{\delta}(\gamma^\mu)_{\dot{\delta}}{}^{\gamma}\\
&+\frac{1}{N_c}\lambda_p(p_1,-p_3,-p_2,p_4)\delta^a_b\delta^c_d\delta^j_i\delta^l_k(\gamma_\mu)_{\dot{\gamma}}{}^{\gamma}(\gamma^\mu)_{\dot{\delta}}{}^{\delta}\\
&+2\lambda_n(p_1,p_2,p_3,p_4)\delta^a_d\delta^c_b(T_z)_i{}^{l}(T^z)_k{}^{j}(\not{p}_1-\not{p}_3)_{\dot{\gamma}}{}^{\delta}(\not{p}_2-\not{p}_4)_{\dot{\delta}}{}^{\gamma}\\
&+2\lambda_n(p_1,-p_3,-p_2,p_4)\delta^a_b\delta^c_d(T_z)_i{}^{j}(T^z)_k{}^{l}(\not{p}_1+\not{p}_2)_{\dot{\gamma}}{}^{\gamma}(\not{p}_3+\not{p}_4)_{\dot{\delta}}{}^{\delta}
\end{aligned}
\tag{E.15}
$$

From there we obtain

$$\gamma_\psi^{(2)} = -\frac{1}{2} Z_\psi \int \frac{d^dp}{(2\pi)^d} \frac{d^dq}{(2\pi)^d} \left(\frac{\partial}{\partial t} r_k(q) - \eta_\psi r_k(q) \right) \cdot$$
$$\sum_a [(c_a(q) + r_k(q))^2 q^2 + m_a^2(q)]^{-2}$$
$$\Big[\bar\psi_a^i(p) \Big\{ -2^{\frac{d}{2}+2} N_c m_a(q)(c_a(q) + r_k(q)) q^2 \bar\gamma \lambda_\sigma(-q, q, -p, p)$$
$$-2^{\frac{d}{2}} N_c [(c_a(q) + r_k(q))^2 q^2 - m_a^2(q)] \not{q} \lambda_\rho(-q, q, -p, p)$$
$$+\frac{2}{N_c} \Big[\Big(2^{\frac{d}{2}} - 2\Big) \Big((c_a(q) + r_k(q))^2 q^2 - m_a^2(q) \Big) \not{q} \tag{E.16}$$
$$+2^{\frac{d}{2}+1} m_a(q)(c_a(q) + r_k(q)) q^2 \bar\gamma \Big] \lambda_p(-q, q - p, p)$$
$$+\frac{2(N_c^2 - 1)}{N_c} \quad \Big[((c_a(q) + r_k(q))^2 q^2 - m_a^2(q))((p^2 - q^2)\not{q} + 2q^2 \not{p} - 2(pq)\not{p})$$
$$+2m_a(q)(c_a(q) + r_k(q)) q^2 (q - p)^2 \bar\gamma \Big] \lambda_n(-q, q, -p, p) \Big\} \psi_i^a(p)$$
$$+[(c_a(q) + r_k(q))^2 q^2 - m_a^2(q)] \bar\psi_b^i(p) \not{q} \psi_i^b(p)$$
$$\Big\{ -4\lambda_\sigma(-q, p, -q, p) + 2(2 - 2^{\frac{d}{2}}) \lambda_\rho(-q, p, -q, p) + 2^{\frac{d}{2}+1} \lambda_p(-q, p, -q, p) \Big\} \Big]$$

For an investigation of $\gamma_\psi^{(4)}$ we first consider the simple truncation (4.25) which is relevant for sect. 4. This corresponds to

$$\Big(G^{(4)}\Big)^{acjl\gamma\delta}_{bdik\dot\gamma\dot\delta}(p_1, p_2, p_3, p_4) = 2Z_\psi^2$$
$$\Big\{ \lambda_m(p_1, p_2, p_3, p_4) \delta_a^d \delta_c^b (T^z)_i{}^l (T_z)_k{}^j (\gamma_\mu)_{\dot\gamma}{}^{\dot\delta} (\gamma^\mu)_{\dot\delta}{}^\gamma$$
$$-\lambda_m(p_1, -p_3, -p_2, p_4) \delta_a^b \delta_c^d (T^z)_i{}^j (T_z)_k{}^l (\gamma_\mu)_{\dot\gamma}{}^\gamma (\gamma^\mu)_{\dot\delta}{}^\delta \Big\} \tag{E.17}$$

We also specialize to massless quarks and $c_a(q) = c(q)$. Furthermore, we will omit from now on the second term in the bracket in (E.12). (The index structure of this term is different and its contribution to $\gamma_\psi^{(4)}$ can be computed in a similar manner.) One obtains

$$\gamma_\psi^{(4)} = Z_\psi^2 \int \frac{d^dp_1}{(2\pi)^d} \cdots \frac{d^dp_4}{(2\pi)^d} \frac{d^dq}{(2\pi)^d} (2\pi)^d \delta(p_1 + p_2 - p_3 - p_4)$$
$$\Big[2^{\frac{d}{2}+1} N_F \lambda_m(-q, -p_3, -q + p_1 - p_3, -p_1) \lambda_m(-q - p_2 + p_4, p_2, -q, p_4) \tilde H(q)$$
$$\tilde h(q - p_1 + p_3)(2q_\mu q_\nu - (p_1 - p_3)_\mu q_\nu - (p_1 - p_3)_\nu q_\mu + ((p_1 q) - (p_3 q) - q^2)\delta_{\mu\nu})$$
$$\{\bar\psi(-p_1)\gamma^\mu T_z \psi(-p_3)\}\{\bar\psi(p_4)\gamma^\nu T^z \psi(p_2)\}$$
$$-4\lambda_m(-q, q - p_1 - p_2, -p_2, -p_1) \lambda_m(-q + p_3 + p_4, q, p_3, p_4)$$
$$\{\bar\psi(-p_1)\gamma^\mu \tilde h(q - p_1 - p_2)(\not{q} - \not{p}_1 - \not{p}_2)\gamma_\nu T_z T^y \psi(-p_3)\}$$

$$\{\bar{\psi}(p_4)\gamma^\nu \tilde{H}(q)\not{q}\gamma_\mu T_y T^z \psi(p_2)\}$$
$$-4\lambda_m(-q, q-p_1+p_3, p_3, -p_1)\lambda_m(-q-p_2+p_4, p_2, -q, p_4)$$
$$\{\bar{\psi}(-p_1)\gamma^\nu \tilde{h}(q-p_1+p_3)(\not{q}-\not{p}_1+\not{p}_3)\gamma_\mu \tilde{H}(q)\not{q}\gamma_\nu T_y T^z T^y \psi(-p_3)\}$$
$$\{\bar{\psi}(p_4)\gamma^\mu T_z \psi(p_2)\}$$
$$-4\lambda_m(-q, -p_3, -q+p_1-p_3, -p_1)\lambda_m(-q-p_2+p_4, q, -p_2, p_4)$$
$$\{\bar{\psi}(-p_1)\gamma^\mu T_z \psi(-p_3)\}$$
$$\{\bar{\psi}(p_4)\gamma^\nu \tilde{H}(q)\not{q}\gamma_\mu \tilde{h}(q-p_1+p_3)(\not{q}-\not{p}_1+\not{p}_3)\gamma_\nu T_y T^z T^y \psi(p_2)\}\Big] \tag{E.18}$$

where N_F is the number of quark flavours and

$$\begin{aligned}
&H(q) = \tilde{H}(q)\not{q}, \quad h(q) = \tilde{h}(q)\not{q} \\
&\tilde{h}^{-1}(q) = (c(q) + r_k(q))q^2 \\
&\tilde{H}(q) = q^2 \tilde{h}^2(q)\left(\frac{\partial}{\partial t} r_k(q) - \eta_\psi r_k(q)\right) = -\tilde{\partial}_t \tilde{h}(q)
\end{aligned} \tag{E.19}$$

For the more general ansatz (E.12) the contribution to the flow equation of the four-point function is rather lengthy and may be split into different terms

$$\begin{aligned}
\gamma_\psi^{(4)} = Z_\psi^2 \int \frac{d^d p_1}{(2\pi)^d} \cdots \frac{d^d p_4}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} (2\pi)^d \delta(p_1+p_2-p_3-p_4) \\
\{A_{\sigma\sigma} + A_{\sigma\rho} + A_{\sigma p} + A_{\sigma n} + A_{\rho\rho} + A_{\rho p} + A_{\rho n} + A_{pp} + A_{pn} + A_{nn}\}
\end{aligned} \tag{E.20}$$

with $A_{\sigma\rho} \sim \lambda_\sigma \lambda_\rho$ etc. As an example we give (again omitting the second term in the bracket in (E.12))

$$A_{\sigma\sigma} = N_c \lambda_\sigma(-q, q-p_1-p_2, -p_2, -p_1)\lambda_\sigma(-q+p_3+p_4, q, p_3, p_4)$$
$$\sum_{a,b}\Big[\{\bar{\psi}_a^i(-p_1)\psi_i^b(p_2)\}\{\bar{\psi}_b^j(p_4)\psi_j^a(-p_3)\} tr(h^{(a)}(q-p_1-p_2)H^{(b)}(q))$$
$$+\{\bar{\psi}_a^i(-p_1)\bar{\gamma}\psi_i^b(p_2)\}\{\bar{\psi}_b^j(p_4)\bar{\gamma}\psi_j^a(-p_3)\} tr(h^{(a)}(q-p_1-p_2)\bar{\gamma}H^{(b)}(q)\bar{\gamma})$$
$$-\{\bar{\psi}_a^i(-p_1)\psi_i^b(p_2)\}\{\bar{\psi}_b^j(p_4)\bar{\gamma}\psi_j^a(-p_3)\} tr(h^{(a)}(q-p_1-p_2)\bar{\gamma}H^{(b)}(q))$$
$$-\{\bar{\psi}_a^i(-p_1)\bar{\gamma}\psi_i^b(p_2)\}\{\bar{\psi}_b^j(p_4)\psi_j^a(-p_3)\} tr(h^{(a)}(q-p_1-p_2)H^{(b)}(q)\bar{\gamma})\Big]$$
$$-\lambda_\sigma(-q, p_2, -q+p_1+p_2, -p_1)\lambda_\sigma(-q+p_3+p_4, -p_3, -q, p_4)$$
$$\sum_c\Big[\left\{\bar{\psi}_a^i(-p_1)h^{(c)}(q-p_1-p_2)\psi_i^b(-p_3)\right\}\left\{\bar{\psi}_b^j(p_4)H^{(c)}(q)\psi_j^a(p_2)\right\}$$
$$+\left\{\bar{\psi}_a^i(-p_1)\bar{\gamma}h^{(c)}(q-p_1-p_2)\bar{\gamma}\psi_i^b(-p_3)\right\}\left\{\bar{\psi}_b^j(p_4)\bar{\gamma}H^{(c)}(q)\bar{\gamma}\psi_j^a(p_2)\right\}$$
$$-\left\{\bar{\psi}_a^i(-p_1)h^{(c)}(q-p_1-p_2)\bar{\gamma}\psi_i^b(-p_3)\right\}\left\{\bar{\psi}_b^j(p_4)\bar{\gamma}H^{(c)}(q)\psi_j^a(p_2)\right\}$$

$$
\begin{aligned}
&-\left\{\bar\psi^i_a(-p_1)\bar\gamma h^{(c)}(q-p_1-p_2)\psi^b_i(-p_3)\right\}\left\{\bar\psi^j_b(p_4)H^{(c)}(q)\bar\gamma\psi^a_j(p_2)\right\}\Big]\\
&-\lambda_\sigma(-q,q-p_1-p_2,-p_2,-p_1)\lambda_\sigma(-q+p_3+p_4,-p_3,-q,-p_4)\\
&\sum_a\Big[\left\{\bar\psi^i_a(-p_1)\psi^a_i(p_2)\right\}\left\{\bar\psi^j_b(p_4)H^{(a)}(q)h^{(a)}(q-p_1-p_2)\psi^b_j(-p_3)\right\}\\
&+\left\{\bar\psi^i_a(-p_1)\bar\gamma\psi^a_i(p_2)\right\}\left\{\bar\psi^j_b(p_4)\bar\gamma H^{(a)}(q)\bar\gamma h^{(a)}(q-p_1-p_2)\bar\gamma\psi^b_j(-p_3)\right\}\\
&-\left\{\bar\psi^i_a(-p_1)\psi^a_i(p_2)\right\}\left\{\bar\psi^j_b(p_4)\bar\gamma H^{(a)}(q)h^{(a)}(q-p_1-p_2)\bar\gamma\psi^b_j(-p_3)\right\}\\
&-\left\{\bar\psi^i_a(-p_1)\bar\gamma\psi^a_i(p_2)\right\}\left\{\bar\psi^j_b(p_4)H^{(a)}(q)\bar\gamma h^{(a)}(q-p_1-p_2)\psi^b_j(-p_3)\right\}\Big]\\
&-\lambda_\sigma(-q,p_2,-q+p_1+p_2,-p_1)\lambda_\sigma(-q+p_3+p_4,q,p_3,p_4)\\
&\sum_b\Big[\left\{\bar\psi^i_a(-p_1)h^{(b)}(q-p_1-p_2)H^{(b)}(q)\psi^a_i(p_2)\right\}\left\{\bar\psi^j_b(p_4)\psi^b_j(-p_3)\right\}\\
&+\left\{\bar\psi^i_a(-p_1)\bar\gamma h^{(b)}(q-p_1-p_2)\bar\gamma H^{(b)}(q)\bar\gamma\psi^a_i(p_2)\right\}\left\{\bar\psi^j_b(p_4)\bar\gamma\psi^b_j(-p_3)\right\}\\
&-\left\{\bar\psi^i_a(-p_1)h^{(b)}(q-p_1-p_2)\bar\gamma H^{(b)}(q)\psi^a_i(p_2)\right\}\left\{\bar\psi^j_b(p_4)\bar\gamma\psi^b_j(-p_3)\right\}\\
&-\left\{\bar\psi^i_a(-p_1)\bar\gamma h^{(b)}(q-p_1-p_2)H^{(b)}(q)\bar\gamma\psi^a_i(p_2)\right\}\left\{\bar\psi^j_b(p_4)\psi^b_j(-p_3)\right\}\Big] \qquad \text{(E.21)}
\end{aligned}
$$

Here the trace tr is over Dirac spinor indices. Other terms like $A_{\sigma\rho}$ and $A_{\rho\rho}$ can be obtained by inserting γ_μ in appropriate places and accounting for appropriate factors 1/2 or 1/4. In the chiral limit these expressions simplify since for $m_a = 0$, $c_a(q) = c(q)$ one has $h(q) = \tilde h(q)\not q$ and $H(q) = \tilde H(q)\not q$. One finds

$$
\begin{aligned}
A_{\sigma\sigma} &= -2^{\frac{d}{2}+1}N_c\frac{q^2-(p_1q)-(p_2q)}{q^2(q-p_1-p_2)^2}\left(\frac{\partial}{\partial t}r_k(q)-\eta_\psi r_k(q)\right)\\
&[c(q-p_1-p_2)+r_k(q-p_1-p_2)]^{-1}[c(q)+r_k(q)]^{-2}\\
&\lambda_\sigma(-q,q-p_1-p_2,-p_2,-p_1)\lambda_\sigma(-q+p_3+p_4,q,p_3,p_4)\mathcal{M}_\sigma(p_1,p_2,p_3,p_4)\\
&-2N_F\lambda_\sigma(-q,-p_3,-q+p_1-p_3,-p_1)\lambda_\sigma(-q-p_2+p_4,p_2,-q,p_4)\\
&\Big[\left\{\bar\psi^i_a(-p_1)h(q-p_1+p_3)\psi^b_i(p_2)\right\}\left\{\bar\psi^j_b(p_4)H(q)\psi^a_j(-p_3)\right\}\\
&+\left\{\bar\psi^i_a(-p_1)h(q-p_1+p_3)\bar\gamma\psi^b_i(p_2)\right\}\left\{\bar\psi^j_b(p_4)H(q)\bar\gamma\psi^a_j(-p_3)\right\}\Big] \qquad \text{(E.22)}
\end{aligned}
$$

where we have made for the second term a replacement $p_2 \leftrightarrow -p_3$ consistent with (E.21). The first term gives a contribution to the flow of λ_σ whereas the second term, upon performing the q-integration, contributes to the evolution of λ_ρ. We observe that the omitted term in $A_{\sigma\sigma}$ arising from the second term in the bracket in (E.12) contributes only to the flow of λ_p.

Appendix F: Evolution equation for the gauge field propagator

In this appendix we compute the evolution equation for the gauge field propagator in a pure Yang Mills theory without fermions. We use the gauge fixing

$$S_{\mathrm{gf}} = \frac{1}{2\alpha}\int d^dx(\partial^\mu A^z_\mu)^2 \tag{F.1}$$

for arbitrary α. We start with the gauge field-dependent inverse propagator in momentum space

$$\begin{aligned}
&(\tilde{\Gamma}^{(2)}_k)^{y\mu}_{\nu z}(q',q) = \tilde{Z}_F P^\mu_\nu(q)\delta^y_z(2\pi)^d\delta(q'-q)\\
&+i\tilde{g}\tilde{Z}_F f_z{}^{yw}\{(2q-q')_\nu A^\mu_w(q-q')\\
&+(2q'-q)^\mu A_{w\nu}(q-q') - (q+q')_\sigma A^\sigma_w(q-q')\delta^\mu_\nu\}\\
&+\frac{1}{2}\tilde{g}^2\tilde{Z}_F\int\frac{d^dp}{(2\pi)^d}\{(f_{zx}{}^u f^y{}_{wu} + f_{zw}{}^u f^y{}_{xu}))\delta^\mu_\nu A^x_\sigma(q-q'+p)A^{w\sigma}(-p)\\
&+2(f_z{}^{yu}f_{xwu} + f_{zw}{}^u f_x{}^y{}_u)A^{x\mu}(q-q'+p)A^w_\nu(-p)\}
\end{aligned} \tag{F.2}$$

Here $P^\mu_\nu(q)$ is the general kinetic operator

$$P^\mu_\nu(q) = g(q)\delta^\mu_\nu + h(q)q_\nu q^\mu \tag{F.3}$$

with g and h functions of q^2. In the classical approximation one would have

$$g(q) = q^2, \quad h(q) = (\frac{1}{\alpha}-1) \tag{F.4}$$

Here we incorporate an infrared cutoff $\sim k$ such that for massless gluons

$$\lim_{q^2\to 0} g(q) \sim k^2, \quad \lim_{q^2\to 0} h(q) \sim \frac{k^2}{q^2}. \tag{F.5}$$

For the present calculation we keep g and h as free functions. Whereas the A-dependent terms in (F.2) can be derived from a term in the action $\sim \frac{1}{4}\tilde{Z}_F F^z_{\mu\nu}F^{\mu\nu}_z$, we consider here first the most general form of the A-independent term.

The evolution equation for the k-dependent effective action for gluons ($\Gamma^{(A)}_k = \Gamma_k[A,\bar{A}=0]$)

$$\frac{\partial}{\partial t}\Gamma^{(A)}_k = \frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\left\{\frac{\partial}{\partial t}[R_k(q)]^\nu_\mu[(\tilde{\Gamma}^{(2)}_k)^{-1}]^{z\mu}_{\nu z}(q,q)\right\} - \epsilon = \gamma_A - \epsilon \tag{F.6}$$

contains contributions from the gauge-field fluctuations (γ_A) and the ghosts (ϵ). They can be identified with corresponding contributions in the flow equation (4.18) for the quark-effective action in the limit of a truncation where $(\tilde{\Gamma}_k^{(2)})^{\alpha}{}_{\alpha'}$ does not explicitly depend on ψ. Defining formally a partial derivative $\tilde{\partial}_t$ acting only on P^{μ}_{ν} with

$$\tilde{\partial}_t P^{\mu}_{\nu}(q) = \tilde{Z}_F^{-1} \frac{\partial}{\partial t} \left(R_k(q) \delta^{\mu}_{\nu} + \tilde{R}_k(q) q_{\nu} q^{\mu} \right) \tag{F.7}$$

we can write γ_A as the derivative of a one-loop expression

$$\gamma_A = \frac{1}{2} Tr \tilde{\partial}_t \ln \tilde{\Gamma}_k^{(2)} \tag{F.8}$$

The evolution of the gauge-field propagator can be extracted from the term quadratic in A in γ_A

$$\gamma_A^{(2)} = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \left\{ V_A(p) A^{\mu}_z(-p) A^z_{\mu}(p) + W_A(p) p_{\nu} A^{\nu}_z(-p) p^{\mu} A^z_{\mu}(p) \right\} \tag{F.9}$$

Our aim is the computation of the functions V_A and W_A which depend on p^2. The ghost contributions will be added later.

We expand

$$Tr \tilde{\partial}_t \ln(\tilde{P} + \Delta) = Tr \tilde{\partial}_t \ln \tilde{P} + Tr \tilde{\partial}_t (\Delta \tilde{P}^{-1}) - \frac{1}{2} Tr \tilde{\partial}_t (\Delta \tilde{P}^{-1} \Delta \tilde{P}^{-1}) + ... \tag{F.10}$$

where Δ is the A-dependent piece in $\tilde{\Gamma}_k^{(2)}$ (F.2). The first term amounts to an irrelevant constant and the contribution linear in A from the second term vanishes due to $f_z{}^{zw} = 0$. We obtain more explicitly

$$\begin{aligned} \gamma_A^{(2)} &= \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \tilde{\partial}_t \left\{ \Delta^{z\mu}_{\nu z}(q,q) \tilde{Z}_F^{-1} (P^{-1})^{\nu}_{\mu}(q) \right\} \\ &- \frac{1}{4} \int \frac{d^d q}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} \tilde{\partial}_t \left\{ \Delta^{y\mu}_{\nu z}(q,p) \tilde{Z}_F^{-1} (P^{-1})^{\rho}_{\mu}(p) \Delta^{z\sigma}_{\rho y}(p,q) \tilde{Z}_F^{-1} (P^{-1})^{\nu}_{\sigma}(q) \right\} \end{aligned} \tag{F.11}$$

with

$$\begin{aligned} (P^{-1})^{\nu}_{\mu}(q) &= \frac{1}{g(p)} \left(\delta^{\nu}_{\mu} - \frac{h(q)}{g(q) + h(q) q^2} q^{\nu} q_{\mu} \right) \\ &= \frac{1}{g(q)} \left(\delta^{\nu}_{\mu} - b(q) \frac{q^{\nu} q_{\mu}}{q^2} \right) \end{aligned} \tag{F.12}$$

or

$$b(q) = \frac{h(q) q^2}{g(q) + h(q) q^2} \tag{F.13}$$

Using the identity[12]

$$f_{zyw}f^{zyx} = (T_zT^z)_w^{\ x} = C\delta_w^x = N_c\delta_w^x \tag{F.14}$$

this yields

$$\begin{aligned}
&\gamma_A^{(2)} = \frac{1}{2}C\tilde{g}^2\int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\tilde{\partial}_t\Big\{g^{-1}(q)\Big[(d-1-b(q))\ A_\mu^z(p)A_z^\mu(-p)\\
&+\frac{b(q)}{q^2}q^\mu q_\nu A_\mu^z(p)A_z^\nu(-p)\Big]\\
&-\frac{1}{2}g^{-1}(q)g^{-1}(q+p)\Big[(5p^2+2(pq)+2q^2)A_\mu^z(p)A_z^\mu(-p)\\
&+((4d-6)q^\mu q_\nu+(2d-3)(q^\mu p_\nu+p^\mu q_\nu)+(d-6)p^\mu p_\nu)A_\mu^z(p)A_z^\nu(-p)\\
&-2\frac{b(q)}{q^2}\Big\{(q^2+2(pq))^2A_z^\mu(p)A_\mu^z(-p)\\
&+[(p^2-2(pq)-q^2)q^\mu q_\nu-(q^2+3(pq))(q^\mu p_\nu+p^\mu q_\nu)+q^2p^\mu p_\nu]A_\mu^z(p)A_z^\nu(-p)\Big\}\\
&+\frac{b(q)}{q^2}\frac{b(q+p)}{(q+p)^2}\Big[(p^2)^2q^\mu q_\nu-p^2(pq)(q^\mu p_\nu+p^\mu q_\nu)\ +(pq)^2p^\mu p_\nu\Big]\ A_\mu^z(p)A_z^\nu(-p)\Big]\Big\}
\end{aligned} \tag{F.15}$$

The term $\sim\ A_\mu^z(p)A_z^\nu(-p)$ can be projected on contributions $\sim\ G_A$ using as a projector $\frac{1}{d-1}\left(\delta_\mu^\nu-\frac{p_\mu p^\nu}{p^2}\right)$, and similarly for contributions $\sim\ H_A$ with the projector $\frac{1}{d-1}\frac{1}{p^2}\left(d\frac{p_\mu p^\nu}{p^2}-\delta_\mu^\nu\right)$. With

$$\begin{aligned}
V_A(p) &= C\tilde{g}^2\int\frac{d^dq}{(2\pi)^d}\tilde{\partial}_t v_A(p,q)\\
W_A(p) &= C\tilde{g}^2\int\frac{d^dq}{(2\pi)^d}\tilde{\partial}_t w_A(p,q)
\end{aligned} \tag{F.16}$$

we obtain

$$\begin{aligned}
&v_A(p,q) = g^{-1}(q)\left[d-1-b(q)+\frac{b(q)}{d-1}\left(1-\frac{(pq)^2}{q^2p^2}\right)\right]\\
&-\frac{1}{2}g^{-1}(q)g^{-1}(q+p)\left[5p^2+2(pq)+2q^2+\frac{4d-6}{d-1}\left(q^2-\frac{(pq)^2}{p^2}\right)\right.\\
&-\frac{2}{d-1}\frac{b(q)}{q^2}\Big\{(d-2)(q^2)^2+(4d-6)q^2(pq)+q^2p^2+(4d-5)(pq)^2\\
&\left.+2\frac{(pq)^3}{p^2}+\frac{(pq)^2q^2}{p^2}\Big\}+\frac{1}{d-1}\frac{b(q)}{q^2}\frac{b(q+p)}{(q+p)^2}\Big\{(p^2)^2q^2-(pq)^2p^2\Big\}\right]
\end{aligned} \tag{F.17}$$

[12]The value $C=N_c$ obtains for the gauge group $SU(N_c)$. We keep in this appendix the Casimir operator C for general nonabelian simple gauge groups.

and

$$
\begin{aligned}
& w_A(p,q) = g^{-1}(q)\frac{b(q)}{q^2}\left[\frac{d}{d-1}\frac{(pq)^2}{(p^2)^2} - \frac{1}{d-1}\frac{q^2}{p^2}\right] \\
& -\frac{1}{2}g^{-1}(q)g^{-1}(q+p)\left[\frac{4d-6}{d-1}\left(d\frac{(pq)^2}{(p^2)^2} - \frac{q^2}{p^2}\right) + (4d-6)\frac{(pq)}{p^2} + d - 6\right. \\
& -\frac{2b(q)}{q^2}\left\{\frac{d-2}{d-1}q^2 - 2\frac{d-2}{d-1}\frac{q^2(pq)}{p^2}\right. \\
& \left. -\frac{5d-6}{d-1}\frac{(pq)^2}{p^2} + \frac{1}{d-1}\frac{(q^2)^2}{p^2} - \frac{d}{d-1}\frac{(pq)^2q^2}{(p^2)^2} - \frac{2d}{d-1}\frac{(pq)^3}{(p^2)^2}\right\} \\
& \left. +\frac{1}{d-1}\frac{b(q)}{q^2}\frac{b(q+p)}{(q+p)^2}\{(pq)^2 - q^2p^2\}\right] \qquad \text{(F.18)}
\end{aligned}
$$

We next evaluate the contribution from the ghosts given by ϵ (3.17). With the approximation (3.18) and $\bar{A} = 0$ one has

$$
\epsilon = Tr\left\{\left(\frac{\partial}{\partial t}R_k^{(\mathrm{gh})}\right)\left(-\partial^\mu D_\mu[A] + R_k^{(\mathrm{gh})}\right)^{-1}\right\} \qquad \text{(F.19)}
$$

where $R^{(\mathrm{gh})}$ is now a function of the normal Laplacian. We generalize this formula by taking into account the most general effective ghost propagator. This corresponds in momentum space to the truncation

$$
\left[\Gamma_k^{(\mathrm{gh})(2)} + R_k^{(\mathrm{gh})}\right]^y_{\ z}(q',q) = Z_{\mathrm{gh}}P_{\mathrm{gh}}(q)\delta_z^y(2\pi)^d\delta(q-q') - i\tilde{g}Z_{\mathrm{gh}}f_{w\ z}^{\ y}A_\mu^w(q'-q)q'^\mu \qquad \text{(F.20)}
$$

An expansion of the propagator up to quadratic order in A gives

$$
\begin{aligned}
& \left[\left(\Gamma_k^{(\mathrm{gh})(2)} + R_k^{(\mathrm{gh})}\right)^{-1}\right]^y_{\ z}(q',q) = Z_{\mathrm{gh}}^{-1}P_{\mathrm{gh}}^{-1}(q)\delta_z^y(2\pi)^d\delta(q-q') \\
& +i\tilde{g}Z_{\mathrm{gh}}^{-1}P_{\mathrm{gh}}^{-1}(q')P_{\mathrm{gh}}^{-1}(q)f_{w\ z}^{\ y}q'^\mu A_\mu^w(q'-q) \\
& -\tilde{g}^2Z_{\mathrm{gh}}^{-1}P_{\mathrm{gh}}^{-1}(q')P_{\mathrm{gh}}^{-1}(q)f_{w\ x}^{\ y}f_{v\ z}^{\ x}\cdot \\
& \int\frac{d^dp}{(2\pi)^d}P_{\mathrm{gh}}^{-1}(p)q'^\mu p^\nu A_\mu^w(q'-p)A_\nu^v(p-q) \qquad \text{(F.21)}
\end{aligned}
$$

This yields, up to an irrelevant constant

$$
\begin{aligned}
\epsilon \;=\; & C\tilde{g}^2\int\frac{d^dq}{(2\pi)^d}Z_{\mathrm{gh}}^{-1}\frac{\partial}{\partial t}R_k^{(\mathrm{gh})}(q)P_{\mathrm{gh}}^{-2}(q)\cdot \\
& \int\frac{d^dp}{(2\pi)^d}P_{\mathrm{gh}}^{-1}(q-p)q^\mu(q-p)_\nu A_\mu^z(p)A_z^\nu(-p) \qquad \text{(F.22)}
\end{aligned}
$$

With

$$\epsilon = -\frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\left\{V_{\rm gh}(p)\delta^\mu_\nu + W_{\rm gh}(p)p^\mu p_\nu\right\} A^z_\mu(p)A^\nu_z(-p) \tag{F.23}$$

and defining $g_{\rm gh}$ and $h_{\rm gh}$ in analogy to (F.16) one finds

$$V_{\rm gh} = C\tilde{g}^2\frac{1}{d-1}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\int \frac{d^dq}{(2\pi)^d}\tilde{\partial}_t\left\{P^{-1}_{\rm gh}(q)P^{-1}_{\rm gh}(q+p)q^\mu(q+p)^\nu\right\} \tag{F.24}$$

$$v_{\rm gh} = \frac{1}{d-1}P^{-1}_{\rm gh}(q)P^{-1}_{\rm gh}(q+p)\left(q^2 - \frac{(pq)^2}{p^2}\right) \tag{F.25}$$

$$W_{\rm gh} = C\tilde{g}^2\frac{1}{d-1}\frac{1}{p^2}\left(d\frac{p_\mu p_\nu}{p^2} - \delta_{\mu\nu}\right)\int \frac{d^dq}{(2\pi)^d}\tilde{\partial}_t\left\{P^{-1}_{\rm gh}(q)P^{-1}_{\rm gh}(q+p)q^\mu(q+p)^\nu\right\} \tag{F.26}$$

$$w_{\rm gh} = \frac{1}{d-1}P^{-1}_{\rm gh}(q)P^{-1}_{\rm gh}(q+p)\left[d\frac{(pq)^2}{(p^2)^2} + (d-1)\frac{pq}{p^2} - \frac{q^2}{p^2}\right] \tag{F.27}$$

Combining the contributions from γ_A and ϵ we arrive at the flow equation for the gluon propagator for the pure Yang-Mills theory. It describes the k-dependence of the function $G_A(q)$ and $H_A(q)$ as defined in (5.1) and reads

$$\begin{aligned}\frac{\partial}{\partial t}G_A(p) &= \frac{\partial}{\partial t}(\tilde{Z}_F g(p) - R_k(p)) = V_A(p) + V_{\rm gh}(p)\\ \frac{\partial}{\partial t}H_A(p) &= \frac{\partial}{\partial t}(\tilde{Z}_F h(p) - \tilde{R}_k(p)) = W_A(p) + W_{\rm gh}(p)\end{aligned} \tag{F.28}$$

We observe that V_A and W_A are defined in terms of G_A and H_A through the relation $\Gamma^{(2)}_k(0) = \tilde{\Gamma}^{(2)}_k(0) - R^{(A)}_k,\ G_A = \tilde{Z}_F g - R_k,\ H_A = \tilde{Z}_F h - \tilde{R}_k$. The system (F.28) constitutes therefore a coupled system of partial differential equations for the functions G_A and H_A which depend on two variables, namely k and p^2. It describes the scale-dependence of the gluon propagator under the influence of the three- and four-gluon interactions contained in $(F_{\mu\nu})^2$ and the corresponding ghost contributions.

It is instructive to study first a simplified version of this system where the propagator $\tilde{\Gamma}^{(2)}_k$ is approximated on the r.h.s. by

$$\tilde{Z}_F g(q) = \tilde{Z}_F q^2 + R_k(q) = \tilde{Z}_F P(q),\ h(q) = 0 \tag{F.29}$$

$$P_{\rm gh}(q) = P(q),\ \tilde{\partial}_t P(q) = \frac{\partial}{\partial t}P(q) \tag{F.30}$$

This corresponds to the classical approximation (F.4) wit $\alpha = 1$. It implies $b(q) = 0$ and

$$\begin{aligned}v_A(p,q) &= -\frac{1}{2}P^{-1}(q)P^{-1}(q+p)\left[\frac{6d-8}{d-1}q^2 + 2(pq) + 5p^2 - \frac{4d-6}{d-1}\frac{(pq)^2}{p^2}\right]\\ &\quad +(d-1)P^{-1}(q)\end{aligned} \tag{F.31}$$

$$w_A(p,q) = -\frac{1}{2}P^{-1}(q)P^{-1}(q+p)$$
$$\left[(4d-6)\frac{(pq)}{p^2} + d - 6 + \frac{2d(2d-3)}{d-1}\frac{(pq)^2}{(p^2)^2} - \frac{4d-6}{d-1}\frac{q^2}{p^2}\right] \qquad \text{(F.32)}$$

Combining this with the ghost contribution one obtains

$$\frac{\partial}{\partial t}G_A(p) = (d-1)C\tilde{g}^2 \int \frac{d^dq}{(2\pi)^d}\frac{\partial}{\partial t}P^{-1}(q) \qquad \text{(F.33)}$$
$$-\frac{1}{2}C\tilde{g}^2 \int \frac{d^dq}{(2\pi)^d}\frac{\partial}{\partial t}\left(P^{-1}(q)P^{-1}(q+p)\right)\left[\frac{6d-10}{d-1}q^2 + 2(pq) + 5p^2 - 4\frac{d-2}{d-1}\frac{(pq)^2}{p^2}\right]$$

$$\frac{\partial}{\partial t}H_A(p) \;=\; -\frac{1}{2}C\tilde{g}^2 \int \frac{d^dq}{(2\pi)^d}\frac{\partial}{\partial t}\left(P^{-1}(q)P^{-1}(q+p)\right)\cdot$$
$$\left[d - 6 + 4(d-2)\frac{pq}{p^2} - 4\frac{d-2}{d-1}\frac{q^2}{p^2} + 4d\frac{d-2}{d-1}\frac{(pq)^2}{(p^2)^2}\right] \qquad \text{(F.34)}$$

If we want to study the behaviour of the gluon propagator for small values of the momentum $p^2 \ll k^2$ we can expand the functions G_A and H_A in powers of p^2. For this purpose we expand $P^{-1}(q+p)$ (with $\dot{P}(q) \equiv \frac{\partial}{\partial q^2}P(q)$)

$$P^{-1}(q+p) = P^{-1}(q) - 2(pq)\dot{P}(q)P^{-2}(q)$$
$$-p^2\dot{P}(q)P^{-2}(q) - 2(pq)^2(\ddot{P}(q)P^{-2}(q) - 2\dot{P}^2(q)P^{-3}(q)) + 0(p^3) \qquad \text{(F.35)}$$

and use the identities

$$\int \frac{d^dq}{(2\pi)^d}f(q^2)q_\mu q_\nu = \frac{1}{d}\delta_{\mu\nu}\int \frac{d^dq}{(2\pi)^d}q^2 f(q^2) \qquad \text{(F.36)}$$
$$\int \frac{d^dq}{(2\pi)^d}f(q^2)q_\mu q_\nu q_\rho q_\sigma = \frac{1}{d(d+2)}(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho})\int \frac{d^dq}{(2\pi)^d}q^4 f(q^2)$$

This yields

$$\frac{\partial}{\partial t}G_A(p) = V^{(0)} + V^{(1)}p^2 + O(p^4) \qquad \text{(F.37)}$$

with

$$V^{(0)} = C\tilde{g}^2 \int \frac{d^dq}{(2\pi)^d}\frac{\partial}{\partial t}\left\{(d-1)P^{-1}(q) - \frac{3d-4}{d}q^2P^{-2}(q)\right\} \qquad \text{(F.38)}$$

$$V^{(1)} = -\frac{1}{2}C\tilde{g}^2 \int \frac{d^dq}{(2\pi)^d}\frac{\partial}{\partial t}\left\{5P^{-2}(q) - \frac{6d-4}{d}q^2\dot{P}(q)P^{-3}(q)\right.$$
$$\left.-\frac{4(3d-2)}{d(d+2)}q^4[\ddot{P}(q)P^{-3}(q) - 2\dot{P}^2(q)P^{-4}(q)]\right\} \qquad \text{(F.39)}$$

and

$$\frac{\partial}{\partial t}H_A(0) = W^{(0)} = -\frac{1}{2}C\tilde{g}^2\int\frac{d^dq}{(2\pi)^d}\frac{\partial}{\partial t}\Big\{(d-6)P^{-2}(q) - \frac{8}{d}(d-2)q^2\dot{P}(q)P^{-3}(q)$$
$$-\frac{16(d-2)}{d(d+2)}q^4[\ddot{P}(q)P^{-3}(q) - 2\dot{P}^2(q)P^{-4}(q)]\Big\} \qquad \text{(F.40)}$$

With $x = q^2$ we define the constants l_n^d, m_n^d, v_d by

$$l_n^d = -\frac{1}{2}k^{2n-d}\int_0^\infty dx\; x^{\frac{d}{2}-1}\frac{\partial}{\partial t}P^{-n} \qquad \text{(F.41)}$$

$$m_n^d = -\frac{1}{2}k^{2n-d-2}\int_0^\infty dx\; x^{\frac{d}{2}}\frac{\partial}{\partial t}\left\{\dot{P}^2P^{-n}\right\} \qquad \text{(F.42)}$$

$$v_d^{-1} = 2^{d+1}\pi^{\frac{d}{2}}\Gamma(\frac{d}{2}) \qquad \text{(F.43)}$$

and find

$$V^{(0)} = -4v_dC\tilde{g}^2k^{d-2}\left((d-1)l_1^d - \frac{3d-4}{d}l_2^{d+2}\right) \qquad \text{(F.44)}$$

$$\begin{aligned} V^{(1)} &= v_dC\tilde{g}^2k^{4-d}\left(10l_2^d - \frac{8(3d-2)}{d(d+2)}m_4^{d+2}\right) \\ &\quad + \frac{4(3d-2)}{d(d+2)}v_dC\tilde{g}^2\int dx\frac{d}{dx}\frac{\partial}{\partial t}\left\{x^{\frac{d}{2}+1}\dot{P}P^{-3}\right\} \end{aligned} \qquad \text{(F.45)}$$

$$\begin{aligned} W^{(0)} &= 2v_dC\tilde{g}^2k^{d-4}\left((d-6)l_2^d - \frac{16(d-2)}{d(d+2)}m_4^{d+2}\right) \\ &\quad + 16v_d\frac{(d-2)}{d(d+2)}C\tilde{g}^2\int dx\frac{d}{dx}\frac{\partial}{\partial t}\left\{x^{\frac{d}{2}+1}\dot{P}P^{-3}\right\} \end{aligned} \qquad \text{(F.46)}$$

Using the definition (4.7) for $P(q)$ one easily verifies the identities $l_n^{2n} = 1,\ m_n^{2n-2} = 1$. For $d = 2$ the coefficients $V^{(0)}$ and $W^{(0)}$ vanish and $V^{(1)}$ is proportional to $5l_2^2 - 4m_4^4 = 10\ln 2 - 2$. For $d = 4$ one has $v_4 = 1/32\pi^2$ and

$$\begin{aligned} V^{(0)} &= -\frac{C}{8\pi^2}\tilde{g}^2k^2(3l_1^4 - 2l_2^6) = 0 \\ V^{(1)} &= \frac{10}{3}C\frac{\tilde{g}^2}{16\pi^2} \\ W^{(0)} &= -\frac{10}{3}C\frac{\tilde{g}^2}{16\pi^2} \end{aligned} \qquad \text{(F.47)}$$

A similar expansion for small values of p^2 can be made without the approximations (F.29). One replaces P in (F.41), (F.42) by P_A or P_{gh} and adds a contribution $\sim \partial\ln\tilde{Z}_F/\partial t$ for the t-derivative of the factor $\tilde{Z}_F$ in R_k.

For a study of the behaviour of (F.33), (F.34) for general p it is convenient to introduce in the definition of P an explicit ultraviolet cutoff Λ[13]

$$P_{k,\Lambda}^{-1}(q) = \frac{\exp\left(-\frac{q^2}{\Lambda^2}\right) - \exp\left(-\frac{q^2}{k^2}\right)}{q^2} = \int_{\Lambda^{-2}}^{k^{-2}} d\alpha \exp(-\alpha q^2) \tag{F.48}$$

This permits to perform explicitly the momentum integration

$$\begin{aligned}
&\frac{\partial}{\partial t} G_A(p) = (d-1) C\tilde{g}^2 \frac{\partial}{\partial t} \int_{\Lambda^{-2}}^{k^{-2}} d\alpha \int \frac{d^d q}{(2\pi)^d} \exp(-\alpha q^2) \\
&-\frac{1}{2} C\tilde{g}^2 \frac{\partial}{\partial t} \int_{\Lambda^{-2}}^{k^{-2}} d\alpha \int_{\Lambda^{-2}}^{k^{-2}} d\beta \int \frac{d^d q}{(2\pi)^d} \exp(-\alpha q^2 - \beta(q+p)^2) \\
&\left[\frac{6d-10}{d-1} q^2 + 2(pq) + 5p^2 - 4\frac{d-2}{d-1}\frac{(pq)(pq)}{p^2}\right] \\
&= -2(d-1)(4\pi)^{-\frac{d}{2}} C\tilde{g}^2 k^{d-2} \\
&-\frac{1}{2}(4\pi)^{-\frac{d}{2}} C\tilde{g}^2 \frac{\partial}{\partial t} \int_{\Lambda^{-2}}^{k^{-2}} d\alpha d\beta (\alpha+\beta)^{-\frac{d}{2}}. \\
&\left(\frac{3d-4}{\alpha+\beta} + p^2\left(5 - \frac{2\alpha\beta}{(\alpha+\beta)^2}\right)\right) \exp\left(-\frac{\alpha\beta}{\alpha+\beta} p^2\right)
\end{aligned} \tag{F.49}$$

$$\begin{aligned}
&\frac{\partial}{\partial t} H_A(p) = -\frac{1}{2}(4\pi)^{-\frac{d}{2}} C\tilde{g}^2 \frac{\partial}{\partial t} \int_{\Lambda^{-2}}^{k^{-2}} d\alpha d\beta (\alpha+\beta)^{-\frac{d}{2}} \\
&\left(d - 6 - 4(d-2)\frac{\alpha\beta}{(\alpha+\beta)^2}\right) \exp\left(-\frac{\alpha\beta}{\alpha+\beta} p^2\right)
\end{aligned} \tag{F.50}$$

Using the integral

$$\lim_{\Lambda\to\infty} \frac{d}{dt} \int_{\Lambda^{-2}}^{k^{-2}} d\alpha \int_{\Lambda^{-2}}^{k^{-2}} d\beta (\alpha+\beta)^{-(y+2)} \exp\left(-\frac{\alpha\beta}{\alpha+\beta} p^2\right) = -4k^{2y} \exp\left(-\frac{p^2}{k^2}\right) \hat{J}(y) \tag{F.51}$$

with

$$\begin{aligned}
\hat{J}(y) &= \int_{\frac{1}{2}}^{1} d\gamma \gamma^y \exp\left(\frac{p^2}{k^2}\gamma\right) \\
\hat{J}(y+1) &= k^2 \frac{\partial}{\partial p^2} \hat{J}(y)
\end{aligned} \tag{F.52}$$

and

$$\hat{J}(0) = \frac{k^2}{p^2}\left(\exp\frac{p^2}{k^2} - \exp\frac{p^2}{2k^2}\right)$$

[13] The previous form of $P(q)$ obtains for $\Lambda \to \infty$.

$$\begin{aligned}
\hat{J}(1) &= -\frac{k^4}{(p^2)^2}\left(\exp\frac{p^2}{k^2}-\exp\frac{p^2}{2k^2}\right)+\frac{k^2}{p^2}\left(\exp\frac{p^2}{k^2}-\frac{1}{2}\exp\frac{p^2}{2k^2}\right)\\
\hat{J}(2) &= 2\frac{k^6}{(p^2)^3}\left(\exp\frac{p^2}{k^2}-\exp\frac{p^2}{2k^2}\right)-\frac{2k^4}{(p^2)^2}\left(\exp\frac{p^2}{k^2}-\frac{1}{2}\exp\frac{p^2}{2k^2}\right)\\
&\quad +\frac{k^2}{p^2}\left(\exp\frac{p^2}{k^2}-\frac{1}{4}\exp\frac{p^2}{2k^2}\right) \qquad \text{(F.53)}
\end{aligned}$$

we find

$$\begin{aligned}
\frac{\partial}{\partial t}G_A(p) &= C\tilde{g}^2\mathcal{G}(p)\\
\mathcal{G}(p) &= -2(d-1)(4\pi)^{-\frac{d}{2}}k^{d-2}\\
&\quad +2(4\pi)^{-\frac{d}{2}}k^{d-2}\exp\left(-\frac{p^2}{k^2}\right)\left\{(3d-4)\hat{J}(\frac{d}{2}-1)\right.\\
&\quad \left.+\frac{p^2}{k^2}\left(5\hat{J}\left(\frac{d}{2}-2\right)-2\hat{J}\left(\frac{d}{2}-1\right)+2\hat{J}\left(\frac{d}{2}\right)\right)\right\} \qquad \text{(F.54)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t}H_A(p) &= 2(4\pi)^{-\frac{d}{2}}C\tilde{g}^2k^{d-4}\exp\left(-\frac{p^2}{k^2}\right)\left\{(d-6)\hat{J}\left(\frac{d}{2}-2\right)\right.\\
&\quad \left.-4(d-2)\hat{J}\left(\frac{d}{2}-1\right)+4(d-2)\hat{J}\left(\frac{d}{2}\right)\right\} \qquad \text{(F.55)}
\end{aligned}$$

In particular, this yields for $d=4$

$$\begin{aligned}
\frac{\partial}{\partial t}G_A(p) &= \beta_A(p) \qquad \text{(F.56)}\\
&= \frac{C}{16\pi^2}\tilde{g}^2k^2\left\{4+12\frac{k^2}{p^2}-8\frac{k^4}{(p^2)^2}-\exp\left(-\frac{p^2}{2k^2}\right)\left(9+8\frac{k^2}{p^2}-8\frac{k^4}{(p^2)^2}\right)\right\}
\end{aligned}$$

The qualitative behaviour of $\beta_A(p)$ can easily be understood by expanding for $p^2 \ll 2k^2$

$$\beta_A(p)=\frac{C}{16\pi^2}\tilde{g}^2p^2\left(\frac{10}{3}-\frac{15}{16}\frac{p^2}{k^2}+\frac{79}{480}\left(\frac{p^2}{k^2}\right)^2-\frac{61}{2880}\left(\frac{p^2}{k^2}\right)^3+\frac{29}{13440}\left(\frac{p^2}{k^2}\right)^4-...\right) \qquad \text{(F.57)}$$

whereas for $p^2 \gg 2k^2$ one has

$$\beta_A(p)=\frac{C}{16\pi^2}\tilde{g}^2k^2\left(4+12\frac{k^2}{p^2}-8\left(\frac{k^2}{p^2}\right)^2\right) \qquad \text{(F.58)}$$

It is instructive to write the gluon propagator in terms of momentum-dependent wave function renormalizations Z_t, Z_l for the transversal and longitudinal gluons

and a gluon mass term $\bar{m}_A^2$:

$$G_A(p)\delta_{\mu\nu} + H_A(p)p_\mu p_\nu = \bar{m}_A^2\delta_{\mu\nu} + Z_t(p)(p^2\delta_{\mu\nu} - p_\mu p_\nu) + Z_l(p)p_\mu p_\nu \tag{F.59}$$

$$\begin{aligned}\bar{m}_A^2 &= G_A(0)\\ Z_t(p) &= \frac{G_A(p) - G_A(0)}{p^2}\\ Z_l(p) &= H_A(p) + \frac{G_A(p) - G_A(0)}{p^2}\end{aligned} \tag{F.60}$$

We may now define a k-dependent gauge-fixing parameter $\alpha(k)$ and $\tilde{Z}_F$ by

$$\begin{aligned}\frac{1}{\alpha(k)} &= Z_l(0)\\ \tilde{Z}_F(k) &= Z_t(0)\end{aligned} \tag{F.61}$$

where both definitions make only sense for $Z_l(0)$ and $Z_t(0)$ positive. From (F.47) we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\alpha &= -\frac{1}{\alpha^2}\left(W^{(0)} + V^{(1)}\right) = 0\\ \frac{\partial}{\partial t}\tilde{Z}_F &= V^{(1)} = \frac{10}{3}C\frac{g_k^2}{16\pi^2}\tilde{Z}_F\\ \frac{\partial}{\partial t}\bar{m}_A^2 &= V^{(0)} = 0\end{aligned} \tag{F.62}$$

where we should mention that the vanishing of $V^{(0)}$ is particular to our choice of the cutoff function R_k. Even though $\alpha(k)$ does not depend on k in our approximation, the relevant quantity for the validity of the approximation (F.29) is the ratio $Z_l(0)/Z_t(0)$. The renormalized gauge fixing parameter

$$\alpha_R(k) = \tilde{Z}_F(k)\alpha(k) \tag{F.63}$$

obeys

$$\frac{\partial}{\partial t}\alpha_R = \frac{10}{3}C\frac{g_k^2}{16\pi^2}\alpha_R \tag{F.64}$$

and moves away from $\alpha_R = 1$ appropriate for (F.29). The approximation (F.29) will therefore break down for k in the vicinity of the confinement scale where g_k^2 becomes large. We finally give the running of $\tilde{g}^2$ in terms of the β function for the renormalized gauge coupling g_k^2

$$\frac{\partial}{\partial t}\tilde{g}^2 = \tilde{Z}_F\left(\beta_{g^2} + \frac{10}{3}C\frac{g_k^4}{16\pi^2}\right) = -\frac{C}{4\pi^2}g_k^4\tilde{Z}_F \tag{F.65}$$

For comparison, we have also computed the flow of the gluon propagator for $\alpha = 0$. We start by summarizing the preceding results for arbitrary α.

$$
\begin{aligned}
&\frac{\partial}{\partial t} G_A(q) = N_c g_k^2 \tilde{Z}_F \int \frac{d^4 q'}{(2\pi)^4} \tilde{\partial}_t \Big\{ \left(G_A(q') + R_k(q')\right)^{-1} \tilde{Z}_F \left[3 - \frac{3}{4} b(q')\right] \\
&-\frac{1}{2} \left(G_A(q') + R_k(q')\right)^{-1} \left(G_A(q+q') + R_k(q+q')\right)^{-1} \tilde{Z}_F^2 \\
&\Big[5q^2 + 2(qq') + \frac{16}{3} q'^2 - \frac{10}{3} \frac{(qq')^2}{q^2} \\
&-\frac{2}{3} b(q') \left(2q'^2 + 10(qq') + q^2 + 11 \frac{(qq')^2}{q'^2} + 2 \frac{(qq')^3}{q^2 q'^2} + \frac{(qq')^2}{q^2} \right) \\
&+\frac{1}{3} b(q') b(q+q') \frac{q^2}{(q+q')^2} \left(q^2 - \frac{(qq')^2}{q'^2} \right) \Big] \\
&+\frac{1}{3} P_{\mathrm{gh}}^{-1}(q') P_{\mathrm{gh}}^{-1}(q+q') \left[q'^2 - \frac{(qq')^2}{q^2} \right] \Big\}
\end{aligned}
\tag{F.66}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial t} H_A(q) = -\frac{1}{2} N_c g_k^2 \tilde{Z}_F \int \frac{d^4 q'}{(2\pi)^4} \tilde{\partial}_t \\
&\Big\{ \left(G_A(q') + R_k(q')\right)^{-1} \left(G_A(q+q') + R_k(q+q')\right)^{-1} \tilde{Z}_F^2 \\
&\Big[\frac{40}{3} \frac{(qq')^2}{(q^2)^2} - \frac{10}{3} \frac{q'^2}{q^2} + 10 \frac{(qq')}{q^2} - 2 \\
&-2b(q') \left(\frac{2}{3} - \frac{4}{3} \frac{(qq')}{q^2} - \frac{14}{3} \frac{(qq')^2}{q^2 q'^2} + \frac{1}{3} \frac{q'^2}{q^2} - \frac{4}{3} \frac{(qq')^2}{(q^2)^2} - \frac{8}{3} \frac{(qq')^3}{(q^2)^2 q'^2} \right) \\
&+\frac{1}{3} b(q') b(q+q') \frac{1}{(q+q')^2} \left(\frac{(qq')^2}{q'^2} - q^2 \right) \Big] \\
&-\frac{2}{3} P_{\mathrm{gh}}^{-1}(q') P_{\mathrm{gh}}^{-1}(q+q') \left[3 \frac{(qq')}{q^2} + 4 \frac{(qq')^2}{(q^2)^2} - \frac{q'^2}{q^2} \right] \Big\}
\end{aligned}
\tag{F.67}
$$

where

$$
b(q) = \frac{(H_A(q) + \tilde{R}_k(q)) q^2}{G_A(q) + R_k(q) + (H_A(q) + \tilde{R}_k(q)) q^2}
\tag{F.68}
$$

For $\alpha \to 0$ we observe that H_A diverges $\sim \frac{1}{\alpha}$ and $b(q)$ approaches one. Defining as above a renormalized α_R one sees that $\alpha_R = 0$ is now an infrared stable fixed point[14]. We further observe that for $\alpha_R \to 0$ the function F_2 defined in eq. (5.8) equals $F_1/(p_1 - p_3)^2$ and only the function $G_A(q)$ determines the effective heavy

[14] We choose $\tilde{R}_k q^2 = \left(\frac{1}{\alpha_R} - 1\right) R_k$ such that $b(q) = 1 + 0(\alpha_R)$ for all values of q^2.

four-quark interaction. The flow equation for G_A can be written in a more compact form using

$$P_A(q) = \tilde{Z}_F^{-1}(G_A(q) + R_k(q)) \tag{F.69}$$

and we approximate the ghost part of $\tilde{\Gamma}_k^{(2)}$ by

$$P_{\mathrm{gh}}(q) = P_A(q), \qquad \tilde{\partial}_t P_{\mathrm{gh}}(q) = \tilde{\partial}_t P_A(q). \tag{F.70}$$

This yields the flow euqation for the propagator for $\alpha_R = 0$

$$\begin{aligned}
&\frac{\partial}{\partial t} G_A(q) = N_c g_k^2 \tilde{Z}_F \int \frac{d^4 q'}{(2\pi)^4} \tilde{\partial}_t \Bigg\{ \frac{9}{4} P_A^{-1}(q') \\
&-\frac{1}{6} P_A^{-1}(q') P_A^{-1}(q+q') \Bigg[13q^2 - 14(qq') + 10q'^2 \\
&-10\frac{(qq')^2}{q^2} - 22\frac{(qq')^2}{q'^2} - 4\frac{(qq')^3}{q^2 q'^2} + \frac{q^2}{q'^2}\frac{q^2 q'^2 - (qq')^2}{(q+q')^2} \Bigg] \Bigg\}
\end{aligned} \tag{F.71}$$

The r.h.s. of the flow equation (F.71) involves not only $G_A(q)$ but also the renormalized gauge coupling g_k and the gluon wave function renormalization constant $\tilde{Z}_F$. We approximate $\tilde{Z}_F$ by the coefficient of the q^2 term in G_A. More precisely, we expand for small q^2, $G_A(q) = \bar{m}_A^2 + G_A^{(1)} q^2 + G_A^{(2)} (q^2)^2 + ...$, and identify $\tilde{Z}_F$ with $G_A^{(1)}$. For the flow equation for the renormalized gauge coupling g_k we rely on the fact that the first two coefficients of the β function

$$\begin{aligned}
\frac{\partial g_k^2}{\partial t} &= \beta_{g^2} = -c_1 \frac{g_k^4}{16\pi^2} - c_2 \frac{g_k^6}{(16\pi^2)^2} - ... \\
c_1 &= \frac{22 N_c}{3} \qquad c_2 = \frac{204}{9} N_c^2
\end{aligned} \tag{F.72}$$

are universal In the region of large g_k we may also use nonperturbative estimates of β_{g^2} derived by different methods [7]. An ansatz for β_{g^2} combined with an estimate of $\tilde{\eta}_F$ fixes also the evolution of $\tilde{g}^2$ and provides all information needed for a numerical solution of the flow equation (F.71).

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